IRE Transactions

on INFORMATION THEORY

Journal Devoted to the Theoretical and Experimental Aspects of Information Transmission, Processing and Utilization.

Volume IT-7 JULY, 1961 Number 3

Published Quarterly

In This Issue

PUBLISHED BY THE
Professional Group on Information Theory

IRE Professional Group on Information Theory

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INFORMATION THEORY

IRE Transactions on Information Theory

A Journal Devoted to the Theoretical and Experimental Aspects of Information Transmission, Processing and Utilization

Volume IT-7 | July, 1960 | Number 3

Published Quarterly

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Contributions

Correspondence

Progress in Information Theory in the U.S.A., 1957-1960*

P. ELIAS, A. GILL, R. PRICE, N. ABRAMSON, P. SWERLING, AND L. ZADEH

This is the first in a series of invited tutorial, status and survey papers that will be provided from time to time by the PGIT Committee on Special Papers, whose Chairman is currently L. A. Zadeh. Hopefully these papers will fill a gap that we have long felt existed in our publication program. In the past, there has been no formal method, short of entire Special or Monograph Issues, of providing basic introductory material or surveys of portions of the information theory field.—*The Administrative Committee.*

Introduction

The following report comprises five parts. Part 1 is concerned with contributions centering on Shannon's theory and the theory of coding. Part 2 deals with those results in the theory of random processes which are of relevance to communication problems. Part 3 surveys advances of a basic nature in pattern recognition. Part 4 is concerned primarily with the detection of signals in noise. Part 5 is given over to prediction and filtering, centering on Wiener's theory and its extensions.

PART 1: INFORMATION THEORY AND CODING

P. ELIAS†, FELLOW, IRE

SINCE 1957, there has been considerable progress in the theory of coding messages for transmission over noisy channels. There have been three main directions of advance. First, there has been work on the foundations of the theory. During this time, American mathematicians interested in probability have shown a serious interest in information theory, especially since Feinstein's work (now available in book form [12]), and since the interest shown by Kolmogorov and Khinchin. Second, a great deal of work has been done on error-correcting block codes for noisy binary channels. This work has involved a good deal of modern algebra, and some mathematical algebraists have been joining the communications research workers in attacking these problems. Third, there has been continuing investigation of procedures in which input messages are coded and decoded sequentially rather than in long blocks. This work and the work on binary block codes both have significant practical implications for electrical communications.

Foundations

Shannon's original demonstration of the noisy-channel coding theorem was an existence proof [31]. Given a channel of capacity C bits per second, and a rate of transmission, R bits per second, the transmitter sends sequences of N channel input symbols. The receiver receives sequences of N channel output symbols and decides which input sequence was transmitted, making this decision incorrectly with probability P. What Shannon showed was that for $R < C$, P could be made arbitrarily small by increasing N. The proof was not constructive, and nothing quantitative was said about how rapidly P decreased as a function of N for given R and C. Feinstein [11], [12] showed that P could be bounded by a decaying exponential in N. His proof covered channels with a simple kind of finite memory. While constructive in principle, it could not be used in practice to construct a code with large N. In 1957, Shannon [32] gave a remarkably concise proof based on his original random coding argument, but more detailed and precise; this also gave an exponential bound to P as a function of N, and extended the proof to channels with considerably more complex memory. Blackwell, Breimann, and Thomasian [2] proved the existence theorem for channels with a finite-state memory of a still more general kind. Wolfowitz [40] and Feinstein [13] have also proved converse theorems—the weak converse being that for $R > C$, P cannot approach zero, and the strong converse being that for $R > C$, P must approach 1.

The kind of technique used by Shannon [32] can be extended to obtain upper and lower bounds to the rate of exponential decay of P with N. Earlier work on binary channels had shown that for a considerable range of R less than C, the upper and lower bounds essentially agreed, and the best possible behavior could be uniquely specified. Similar results have been obtained by Shannon for more general channels. This work is not yet published

* This work was supported partially by the US Army Signal Corps., the AF Office of Scientific Research and the Office of Naval Research. The material is part of a Commission 6 progress report prepared for the U.S.A. National Committee of URSI (International Scientific Radio Union) for presentation at the XIII$^{\text{th}}$ General Assembly of URSI in London, Eng., September 5–15, 1960. Text of a report by L. Weinberg on circuit theory, an earlier version of the present one, and reports by other commissions of URSI, may be found in the November-January, 1960, issue of the *J. Res. NBS.*

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t the case of a continuous channel with additive Gaus-n noise has been treated in detail [34].

The increasing interest of mathematicians in this field evidenced by an article by Wolfowitz [39]. In general, e results which the mathematicians have obtained are mer proofs, under more general circumstances, of eorems whose general character was not surprising to mmunications researchers. However, a recent paper [3] s presented an interesting new problem, defining pacity and proving a coding theorem for a channel hose parameters are not known precisely, but are con-rained to lie in known ranges. This work might be levant to incompletely measured and time-varying dio channels. So might a paper by Shannon [33] on annels, in which the transmitter has side information ailable about the state of a channel with memory: example would be the information obtained by measure-ents of the propagation medium obtained while com-unicating.

Binary Channels

Starting with the earlier work of Hamming [16] and epian [35], [36], error-correcting block codes for binary annels have been investigated extensively. Peterson d Fontaine [24] have searched for best possible error-rrecting codes of short block length (up to 29), using computer. The number of codes grows so rapidly with ock length that it was necessary to use many equiva-nce relations and short cut tests to eliminate codes from nsideration early. A number of counter-examples were und to common conjectures about optimum codes.

The use of error-correcting codes, in practice, has been nited by the difficulty of implementation, and by the ct that in many applications of interest, the errors the channel are not independent, but occur in runs or ursts. In an earlier work, Huffman [17] had shown a ding and decoding procedure for the Hamming code hich was simple to implement, and Green and San oucie [14] have shown an easy implementation for a ort multiple-error-correcting code. Hagelbarger [15] as described codes in which correct errors occurr in ursts and whose implementation is not too difficult; bramson [1] has described a highly efficient and easily nplemented set of codes with similar properties.

Work on codes of longer length, which can correct ultiple errors, started with a decoding procedure given y Reed [28] some time ago for the Reed-Muller family of odes. For really large block lengths, these codes are ot efficient; but Perry [23] has built a coder and decoder r a Reed-Muller code which has block length of 128 igits, 64 of which are information digits and 64 check igits. This code can correct any set of 7 or fewer errors mong the group of 128, and the efficiency is quite good. sing microsecond switching devices, the units can keep p with millisecond binary digits.

Calabi and Haefeli [6] have investigated in detail the urst correcting properties of a family of codes which ad been introduced earlier for correction of independent errors [7]. They also discuss the implementation of these codes.

A new family of codes discovered by Bose and Ray-Chaudhuri [4], [5] is much more efficient than the Reed-Muller codes for large block lengths. Although in the limit of infinite block length, these codes may also have zero efficiency, at lengths of a few thousand digits they are still quite good. Peterson [25] has discovered an economical way to decode these codes. There is a great deal of current work on finding more properties of these codes, finding similar codes for channels which are symmetric but not binary, and so forth.

There has been a good deal of recent work on cyclic codes, including some encouraging results on step-by-step decoding due to Prange [27]. Cyclic codes are closely related to the sequences which can be generated by shift registers with feedback connections. Recent discussions of these sequences have been given by Elspas [9] and by Zierler [42]. A review of the recent algebraic work on coding theory, including the Galois field theory which enters in the Bose-Chaudhuri codes, will be given by Peterson in a monograph to be published shortly [26]. Most of the results in this area extend to channels which have an input alphabet of symbols whose number is not 2, but any prime to any power, the channel still being completely symmetric in the way it makes its errors. Nonbinary channels have been investigated in their own right by Lee [20] and by Ulrich [38].

The introduction of two thresholds rather than one in a continuous channel introduces a null zone. The transmitter sends a binary signal, but the receiver makes a ternary decision, not attempting to guess the value of signals received in the null zone. Introducing the null zone may increase channel capacity, as shown by Bloom, *et al.* [30]. It also has the valuable effect of reducing the amount of computation required in decoding, since it is easier to replace missing digits than to correct incorrect ones. This is especially relevant for application to channels with Rayleigh fading.

Sequential Decoding

Earlier work had shown that the block coding procedure could be modified (in the binary case) by constructing codes in a convolutional fashion, so that the coding and decoding of each digit was of the same character and involved the same delay [8]. The parameter which replaces block length in such an argument is the delay between the receipt of a digit and the attempt to decode it reliably. This simplified the coding, but left the decoding procedure as complicated as ever. However, Wozencraft [41] has shown that a suitable sequential-coding procedure may be followed by a sequential-decoding procedure which reduces the average amount of decoding computation immensely. Like the best of the long block codes now in prospect, this procedure promises millisecond communication with microsecond switching circuitry in the decoder at very high reliability. Unlike the block codes, however, Wozencraft's procedure is

statistical and not highly algebraic, and it may be expected to generalize to other discrete channels with no special symmetry properties. On the other hand, the computation remains reasonable only for a range of R well below C. Epstein [10] has studied a sequential decoding procedure for the erasure channel; work on more general channels is under way.

Conclusions on Coding

The general conclusions of interest for applications of error-correcting codes are two. First, there are now several good, small codes which correct bursts of errors, and which could be instrumented fairly easily for use in situations in which a rate well below capacity can be tolerated, so that short codes may be used. These may find early application in sending digital data over telephone lines. Second, there are now available several kinds of large block codes and sequential codes which will permit very reliable transmission over long-distance scatter channels, which can also be implemented. The cost of implementation is appreciable in these cases, but current computer circuitry is fast enough to permit decoding at transmission rates of the order of a thousand binary digits per second, coded in blocks or with sequential constraints hundreds of digits in length; the alternatives of more large antennas or greater transmitter power are also expensive. It seems likely that such systems will be in experimental use by the next international URSI meeting in 1963.

Other Topics

Less progress has been made in the economical coding of information sources. In part, this is because such progress becomes work in speech analysis or, not of television systems and not information theory as such. However, it might be worth noting that a scheme for coding runs of constant intensity in television has been demonstrated at full television speed by Schreibex [29].

A relation between the bandwidth and the duration of a signal is imposed by the Heisenberg uncertainty principle, whose applicability to time functions was pointed out by Gabor many years ago. Kay and Silverman [19] have examined this relationship more carefully, and a form of the uncertainty principle which places a lower bound on the sums of entropies, rather than on the products of second moments, is discussed by Leipnik [21]. Stam [37] also discusses this entropic inequality and closely related results.

The sampling theorem is closely related to these questions. Linden and Abramson [22] have given a generalization which permits the closed form expression of a band-limited function in terms of samples of the function and of its first k derivatives, taken at time intervals $(k + 1)$ times as far apart as is required for samples of the function value alone. This extends earlier work by Jagerman and Fogel [18]. Results bearing both on the uncertainty principle and on approximate sampling theorems, *i.e.*, theorems concerning functions which include all but a fraction δ_1 of their energy in bandwidth W and all but a fraction δ_2 of their energy in a time interval of duration T—are the subject of active current work.

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PART 2: RANDOM PROCESSES

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RESEARCH on random processes in the period under consideration may be conveniently summarized under three main headings: statistical proper-
ies of the output of nonlinear devices, estimation theory
or random processes, and representation theory for
andom processes.

Under the first heading, the investigations concern
he statistical properties of the output of a nonlinear
evice, or of a linear filter following a nonlinear device,
hen the input is a random process having prescribed
atistics. These problems are of great interest since this
a model for many types of receivers. The period 1957–
960, continuing earlier work, has seen the build up of a
rge inventory of results and of methods for attacking
is class of problems.

One of the most comprehensive approaches is reported
n in papers by Darling and Siegert, and by Siegert
]–[3]. These papers, published in 1957 and 1958, report on work actually done earlier. The problem considered is that of finding the (first-order) probability distribution function of the quantity

$$\int \phi[x(\tau), \tau]\, d\tau,$$

where ϕ is a prescribed function, and $x(\tau)$ is a component of a stationary n-dimensional Markoff process. Many problems in the category under consideration are special cases of this. The approach is via the characteristic function of the required probability distribution; it is shown that this characteristic function must satisfy two integral equations. Under certain conditions, it can also be shown that the characteristic function must satisfy two partial differential equations.

Another type of problem in this category is the investigation of the second- or higher-order probability distributions of the output, and particularly of the autocorrelation function of the output or the cross correlation between two or more such outputs. For example, Price in [4] gives a theorem which is useful in deriving such auto- and cross-correlations when the inputs are Gaussian. The theorem stated can be used in many cases to calculate the quantity

$$R = \text{expected value of } \left\{ \prod_{i=1}^{n} f_i(x_i) \right\},$$

where $(x_1, \cdots, x_n)$ is a Gaussian vector and f_i are prescribed functions.

Many other papers, for example [5]–[11], have been written giving special results and using a number of different approaches.

Work has also continued on the problem of the distribution of zero crossings of Gaussian processes [12], [13].

Under the heading of estimation theory for random processes one might first mention the subject of estimating the spectral density of stationary Gaussian processes. Two references, [14] and [15], published in the period 1957–1960, summarize much work on this problem, a great deal of which had been done previously (but not all of which had been published previously). Blackman and Tukey discuss two types of estimates of the power spectrum, *viz.*, estimation of the autocorrelation function, multiplication by a prescribed function of time called a "lag window," followed by Fourier transformation; or passing the observed process through a filter of specified transfer function and calculating the average power of the output. They derive expressions for the first and second moments of such estimates, as well as for the cross moments of estimates of the spectral density at two different frequencies. Grenander and Rosenblatt discuss similar types of spectral estimates, emphasizing and utilizing the fact that these, as well as most other useful estimates of spectral density, are quadratic forms in the observed data. They derive first- and second-order moments, as well as asymptotic probability distribu-

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tions for large observed samples, of such estimates.

A recent paper of Grenander, Pollak, and Slepian [16] discusses the small sample case, relying heavily on the fact that spectral density estimates are usually quadratic forms in the observed data.

In an interesting paper [17] Slepian has discussed the following hypothesis-testing problem: given an observed sample of a Gaussian random process, known to be characterized by either one of two prescribed power spectra, which power spectrum does the process actually have? It turns out that in problems of this type, the measures induced by the two alternative hypotheses may be singular with respect to each other; in which case, it is possible to decide between the alternatives with arbitrarily small error probability, and with an arbitrarily small sample. Slepian gives various sufficient conditions for this. The power spectra satisfying his conditions are, moreover, standard types very frequently postulated. This emphasizes that the mathematical model one chooses must be carefully chosen to be appropriate to the problem one is trying to solve.

Another type of estimation problem for random processes is considered by Swerling [18]. Suppose a prescribed waveform, depending on one or more unknown parameters, is observed in additive Gaussian noise having prescribed autocovariance function and zero mean. Expressions are derived for the greatest lower bound for the variance of estimates of the unknown parameters having prescribed bias. These greatest lower bounds are found to coincide in certain special cases with the variance, obtained by Woodward, of maximum likelihood estimates of the unknown parameters. Similar problems are investigated by Middleton [19].

In the field of representation theory for random processes, work has continued on the subject of representation of nonlinear operations on random processes—especially for Gaussian processes. Papers by Zadeh [20] and Bose [21], and a book by Wiener [22] deal with this problem. The approach followed is, first, to express the initial random process $\{x(t)\}$ as a series

$$x(t) = \sum_{n=1}^{\infty} u_n \alpha_n(t),$$

where $\{\alpha_n(t)\}$ is a set of orthonormal functions over the interval of definition of $\{x(t)\}$. If $\{x(t)\}$ is Gaussian, the u_n are Gaussian and, if $\alpha_m(t)$ are properly chosen, can be made independent. Any linear or nonlinear functional of $\{x(t)\}$ can then be regarded as a function of $u_1, \cdots, u_n, \cdots$. Second, one may choose a set of functions of the variables u_n which are orthonormal in the stochastic sense (as explained, for example, in Zadeh [20] with respect to the process $\{x(t)\}$. Then, nonlinear functionals of $\{x(t)\}$ may be expanded in a series of the orthogonal functions of the variables u_n.

Other research in the field of representation theory has treated such subjects as: Use of bi-orthonormal expansions [11], envelopes of waveforms [23], [24], the sampling theorem and related topics [25], [26], and harmonic analysis of multidimensional processes [27]. Much of this work in representation theory provides useful tools for attacking the problems discussed under the first two headings above.

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PART 3: PATTERN RECOGNITION

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Introduction

PATERN recognition, in its widest sense, cuts across many fields of engineering interest—from signal detection to learning theory, and from me-nical translation to decision-making techniques. Inas-ich as the problem of recognizing patterns is that of nulating human thinking processes, it is also closely ated to nonengineering fields such as physiology, ychology, linguistics and cryptology. No attempt is de in this report to summarize the developments in all ese areas. Rather, pattern recognition developments reported only to the extent that they represent a ntribution to the theory of information. The papers erred to below are primarily those published in Ameri-n engineering journals from 1957 to date; consequently, will be found that the emphasis in this report is placed the recognition of *visual* patterns, rather than vocal, guistic, or other patterns, which are mainly covered in nengineering publications.

The reason for the keen engineering interest in the cognition of visual patterns is the recent emergence of e following two urgent problems: a) How can redundan-s be removed from television pictures, so that video nals could be transmitted at a greatly reduced wave-nd. b) How can printed documents be read automati-lly, so that the most serious bottleneck—the human pist or card puncher—could be eliminated from digital ta-processing systems. Although these two topics are ated separately in the literature, both represent the me general problem of pattern recognition. In the lowing review, this problem will be divided, rather tificially, into the following three phases: 1) redundancy idies, 2) recognition procedures, and 3) learning systems.

Redundancy Studies

Both the compression of television bandwidth and the sign of character recognizers require the determination the source redundancies. The knowledge of these lundancies results in effective recognition criteria, as well as economies in the contemplated recognition system. Powers and Staras [33] suggest to separate picture redundancy into nonpredictive redundancy, resulting from nonoptimal first-order probability distribution, and predictive redundancy, resulting from statistical correlation between successive signals. Experimental work shows that nonpredictive redundancy in television pictures is essentially zero; predictive redundancy permits at least two-to-one saving in bandwidth requirements. Two-to-one saving is also concluded by Deutsch [6], in the case of typewritten or printed alphabetic characters. Kovasznay and Arman [25] propose a new practical method for measuring the autocorrelation function of two-dimensional random patterns; with the aid of this method, the entire function can be obtained at once in the form of a light distribution on a plane.

On a more theoretical level, Gill [13] produces bounds to the number of nonredundant cells in noiseless and noisy patterns, and presents an algorithm for locating these cells. Stearns [37] proposes a method for removing redundancies from given patterns, which is basically a method for reducing Boolean equations containing a large number of "don't care" terms.

A recognition system designed to serve human beings must take into account not only the source characteristics, but also the characteristics of the human "load." Schreiber and Knapp [35] exploit both picture redundancies and human vision limitations to code video signals and to transmit the code at a uniform rate. Graham [16] describes a series of subjective experiments whose purpose is to evaluate the range of transmitted pictures satisfactorily interpreted by human observers.

Several results have been obtained through which the efficiency of automatic pattern recognizers can be compared with the efficiency of human recognizers. Pierce and Karlin [32] report that human beings can transmit printed information, by reading, at the rate of up to 50 bits per second. The accuracy of human recognition of hand-printed characters is found by Neisser and Weene [31] to be less than 97 per cent. Singer [36] concludes that the human recognition process is not limited by the visual channel, whose capacity is 10^{10} bits per second, but by the brain.

Michel [28] shows how the statistical characteristics of the pattern source can be helpful in devising efficient coding schemes for picture transmission. A particular scheme, known as "variable-length coding," is described by Michel, Fleckenstein and Kretzmer [29]; in this scheme, the transmission rate may be made proportional to the source complexity, to result in considerable saving in bandwidth requirements. Capon [4] computes the theoretical bounds on this saving, considering patterns as first-order Markoff processes. Heasly [22] shows how a character-sensing communication channel can be matched to the source to yield the maximum over-all information flow.

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Recognition Procedures

Basically, the process of pattern recognition is that of sequentially sorting a large number of elements into a relatively small number of classes, according to a predetermined set of characteristics. Burge [3] tackles the sorting problem from a general point of view, devising optimal sorting strategies by minimizing appropriate "sorting trees." He concludes that the best strategy depends on the amount of order already existing in the data, where "order" is defined as the minimum amount of work required to sort the data into complete order. Hartmanis [21] develops an algebra of partitions, to facilitate the decomposition of a complex sequential process into several simpler ones; he also formulates the necessary and sufficient conditions for such decomposition. McLachlan [27] proposes a special mathematical discipline, called "description mechanics," for the characterization of general recognition processes. In this proposal, a pattern is a special case of a "description domain," divisible into cells whose size is determined by the prescribed resolution; pattern classes are special cases of "occupant classes," whose number is determined by the prescribed recognition accuracy.

The sorting procedure itself is carried out by searching for various properties in the unknown pattern, and comparing them with the properties of a "reference set" of patterns. Unger [39] describes a system, employing a space-oriented computer, capable of detecting a predetermined set of geometrical properties; association of each reference pattern with a subset of these properties yields the recognition of the unknown pattern. Similar systems are proposed by Bomba [2], who extracts the geometrical properties by operations on small sections of the unknown pattern, by Greenias, Hoppel, Kloomok and Osborne [19], who recognize patterns by the relative size and position of the pattern elements, by Kamentsky [23], who extracts the geometrical properties with neuron-like sensing elements, and by Dimond [8], who processes handwritten characters by subjecting them to special coordinate constraints. Tersoff [38] describes a device which facilitates the property-extraction operations by minimizing the effects of pattern tilt and extraneous marks. Kirsch, Cahn, Ray, and Urban [24] describe laboratory apparatus intended for finding suitable sets of properties for given patterns. The problem of designing logical circuits to carry out the recognition procedure was treated by Evey [10], who proposed various schemes for optimizing this logic.

Glantz [14] formulates a general recognition procedure, employing an "operator" which specifies the method of comparison between the unknown and reference patterns, and a "threshold" which must be overcome for satisfactory recognition. Some of these ideas are carried out by Gold [15] who applies a set of fixed "language rules" and statistically determined threshold tests to recognize hand-sent Morse code.

One criterion for satisfactory recognition, which aroused considerable interest, is the minimum average cost (the Bayes risk) criterion, proposed by Chow [5]. In Chow's recognition system, the patterns signal, the noise statistics, and the cost of misrecognition are known in advance; on the basis of this knowledge the conditional probability of the unknown noisy pattern is computed and weighted with respect to every possible noiseless pattern, and identification is made as to minimize the expected cost. The choice of "noiseless" patterns to be used as reference in this system is discussed by Flores and Grey [11]; they give criteria for optimizing these patterns in the case of white Gaussian noise, and prove that the best pattern coding to be used under such conditions is not necessarily binary. Dickinson [7] describes the application of Chow's system to slit-scan recognition of low-noise and small-size pattern sets. The design of synthetic pattern sets for reference purposes is discussed by Flores and Ragonese [12], who give formulas based on the geometry of the patterns and the empirical properties of the sensing apparatus. Greenias and Hill [18] define measures of character quality and style to aid in the design of synthetic characters.

Learning System

As indicated in the previous section, the prerequisite for the design of an efficient pattern recognizer is the determination of a set of invariants, in terms of which the patterns can be uniquely defined. While many investigators select this set on the basis of intuition and personal experience, others prefer to let a computer make the selection through some "learning" process. Doyle [9] describes a system which collects statistical data on known patterns in order to formulate a series of tests to be used later on unknown patterns. The pattern recognizer proposed by Bledsoe and Brown [1] "learns" the patterns by marking the states of cell pairs randomly distributed over the pattern area. A general learning system, the perceptron, is described by Rosenblatt [34] and Murray [30]; this system, comprising logically simplified neural elements, learns how to discriminate and identify perceptual patterns, after undergoing a special "training" program. Mattson [26] describes a logical system which can adjust itself to realize various pattern processing requirements. Uttley [40] proposes an "inductive inference" machine which can imitate trial-and-error learning by computing conditional probabilities of known patterns.

A different point of view is adopted by Greene [17] whose system memorizes "perceptual units" (Gestalten) such as a circle, a triangle, or a square, in order to identify more complex patterns; the perceptual ability of this system is enhanced by making it obey certain equations of quantum mechanics. Harmon [20] describes a similar system, where the perceptual units are recognized by means of a circular scan.

he learning systems mentioned above, chosen for ir immediate applicability to pattern recognition, are resentative of a much larger number of "artificial elligence" systems, the discussion of which is beyond scope of this report.

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PART 4: DETECTION THEORY

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Introduction

THE PERIOD since the XII General Assembly has seen a consolidation of the closely related concepts of Wald, Woodward, Middleton and Van Meter, and Peterson, Birdsall and Fox into a fairly unified theory of detection, together with the successful application of the theory to a variety of problems. Through this approach, "optimal" detector structures for electronic systems can be synthesized, provided that the designer has *a priori* knowledge of the governing statistics and error costs. At the same time, older and more standard detection techniques have continued to receive attention, the theoretical results generally being stated in terms of probability-of-error or SNR at the detector output.

The maturing of the field of detection theory in the past three years is evidenced by the fact that during this period four books dealing with detection theory (to some extent) were published. The first of these was "Random Signals and Noise," by Davenport and Root [1],

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in which several detection problems of a simple nature were discussed in the last chapter. "Principles and Applications of Random Noise Theory," by Bendat [2] also discussed the detection problem with particular emphasis being given to the errors in various autocorrelation measurements. "Introduction to Statistical Communication Theory," by Middleton [3] is another of these four books dealing with the theory of signal detection. In this comprehensive book, a wealth of specific detection problems are treated and the performance characteristics of many optimum and suboptimum detection systems are calculated. "Statistical Theory of Signal Detection," by Helstrom [4] is a book devoted to the detection problem alone, although Helstrom's definition of detection is broad enough to include the closely related subjects of signal resolution and estimation of signal parameters.

It appears that roughly half the effort of the past three years has been devoted to specific detection problems in radar and communications. In contemporary communications studies, considerable heed is paid to "optimum" detection procedures, there being less inclination to examine conventional, suboptimum detectors than in the radar analyses. The reason for this may be that the radar designer faces considerably greater *a priori* uncertainty, both with regard to the signal and the channel through which it comes. By contrast, relatively simpler channels may usually be assumed without loss of realism in communications problems, and the communications system designer has more direct control of the signal. The appropriate optimum detectors for communications then turn out to be rather elementary, and can at present be constructed with hardly more effort than suboptimum devices require. In fact, the communications environment is generally "clean" enough so that much recent work has been concerned with determining good sets of transmitted signal waveforms, the use of an optimum receiver being taken for granted.

Communications

Some problems of a practical nature associated with the selection of good sets of signals for various types of digital phase-modulation systems are discussed by Cahn [5], Lawton [6], [7] and Hopner [8]. A more general approach to the problem of the selection of signals and the shaping of pulses was given by Sunde [9]. In his paper comparing AM and FM methods of pulse transmission, he concludes that FM has an advantage over AM for the case of a fixed bandwidth channel perturbed by additive white noise.

Reiger [10] has looked at problems of the selection of a set of signals and the use of error-correcting codes. Some simple results seem to indicate that, for small block lengths, if the equipment complexity caused by a large number of signals can be tolerated, a greater improvement may be obtained by use of these waveforms than by use of error-correcting codes.

One example of a communication channel which is not "clean" is the channel with fading. One of the earliest studies concerned with the fading channel was done by Masonson [11]. Masonson analyzes the transmission of binary messages through noise and fading with several types of systems. An analysis of slowly-fading, frequency-nonselective channels perturbed by white Gaussian noise was performed by Turin [12]. Turin obtained general expressions for the error probabilities of such a channel in both the coherent and noncoherent cases, and he applied the results to FSK systems with a variety of pulse shapes. Turin [13] has also examined the selectively fading communication channel and has found that even if one of two independently fading paths is relatively weak, the error probability is considerably lower than if only the stronger path is present. Some particularly important results in the "optimum" (*i.e.*, *a posteriori* probability computer) detection of signals perturbed by a "Gaussian" random channel are given in a paper by Kailath [14]. Kailath shows that the concept of detection by a matched filter (optimum for the case of a known channel) can be generalized in the case of the randomly perturbed channel. The optimum receiver for the randomly perturbed channel is still a matched filter where, however, the "matching" is with respect to a subsidiary estimate of the output of the random channel.

Another method of handling communication problems caused by fading channels is that of diversity reception. Pierce [15] has analyzed the improvement available through diversity reception, and has obtained expressions for the probability of error for both square-law combining and "switch" diversity.

Radar

During the last three years, there have been several analyses of the detection performance of specific types of radar detectors. Cohn and Peach [16] have described equipment for the direct measurement of waveform probabilities. Dilworth and Ackerlind [17] have used Monto Carlo techniques in order to measure output probability distributions of filter-linear detector-integrator and filter-squarer-integrator combinations. Bussgang, Nesbeda and Safran [18] have provided a simplified analysis of sweep-integrator systems containing square-law detectors. Green [19] has analyzed the logarithmic detector and found that it is about 1 db worse than a square-law detector. Stone, Brock and Hammerle [20] have found the probability densities of the output of a filter-squarer-filter detector with constant and with Rayleigh-fading input signals.

Miller and Bernstein [21] have performed an analysis of the first-order effects of interchannel correlation in a bank of filters covering a region of Doppler uncertainty. Their results indicate that, for idealized coherent integrators, the more filters, the better the system performance will be. Some more quantitative results on the effect of interchannel correlations have been obtained by Galejs

l Cowan [22]. They have been able to calculate cor- tions to false alarm and incorrect dismissal probabilities e to noise correlation in contiguous channels.

A general analysis of the radar detection problem, timum detector synthesis, and the evaluation of the rformance of these detectors using orthogonal expansion ordinates, has been given in two reports by Reed, lly and Root [23]. Max [24] has investigated the ssibilities of mismatched filters to combat clutter. He s been able to obtain integral equations whose solutions ld improved performance against clutter.

The problem of detection of random signals in a vari- le strength noise environment is one in which we can pect to see a good deal of work in the future. One study aling with this problem has already been completed Siebert [25]. Siebert discusses a constant false alarm- e detector for use when the noise is of variable strength.

Detection of Stochastic Signals

In the previous two sections devoted to communica- ns and to radar, we have had several occasions to refer work being done on the detection of stochastic signals noise. In this section, we shall mention several other dies in detection of stochastic signals which are not ssified as primarily communications, or primarily radar dies. Strum [26] has discussed the use of microwave liometry for detection with special emphasis to its use radio astronomy. In an appendix, he has shown that uare-law detection is slightly superior to linear (either erage or peak-envelope) detection for low SNR. Kelly, ons and Root [27] have given a more general demon- ration that the square-law is optimum.

Middleton [28], [29] and Kailath [30] have investigated e detection of stochastic signals in noise and have rived at a form of optimum detector which may be nthesized as a time-varying linear filter. Under some- hat more restrictive conditions, Price [31] has shown at the optimum detector may be synthesized as a odified type of radiometer called a "weighted radio- eter," which unites conventional radar notions of pre- d postdetection sweep integration with radiometer inciples. In a paper dealing with the detection of lsed signals in noise, Swerling [32] obtains results for wide variety of signal fading characteristics. The system nsidered consists of a predetection stage, a square-law velope detector, and a linear postdetection integrator. he results obtained are expressions for the Laplace trans- rm of the probability density of the integrator output.

Detection Experiments

The period since the XII General Assembly has seen e success of two radar detection experiments of con- derable importance. In February, 1958, shortly after the ne when Venus and Earth were at close approach, the illstone Hill radar of the Massachusetts Institute of echnology, Lincoln Laboratory, was used in four at- mpts to detect and range Venus [33], [34]. At each attempt, several thousand pulses were emitted, each of 2 msec width and 440 Mc carrier frequency. The transmission lasted for the Earth-Venus-Earth round trip travel time of about 4.5 minutes, and was followed by an equal interval of reception. The received signal was not processed immediately, but was sampled by a crystal-controlled switch and recorded digitally for later processing by an IBM 704 computer programmed as a weighted radiometer.

Members of the Radioscience Laboratory of Stanford University were able to train an array of four rhombic antennas on the Sun for brief periods during April, 1959, and again during September, 1959. Several radar runs were made, each run being a transmission of twelve minutes duration, followed by twelve minutes of reception. The transmission was a sequence of thirty seconds ON, thirty seconds OFF alternations, with a carrier frequency of 25.6 Mc. As with the 1958 Venus Lincoln attempt, the received signal was recorded, and it took nearly a year of analysis before results could be announced [35].

The *A Priori* Problem

Several attempts have been made during the past three years to circumvent the *a priori* problem. Abramson [36] has used some results in the theory of experiment design to show how it is possible to say that one system is superior to another regardless of cost assignments and *a priori* message probabilities. Bellman and Kalaba [37] have employed dynamic programming to study the learning process, and to suggest methods of obtaining *a priori* probabilities when they are not known, or when they are changing. Capon [38] has used nonparametric techniques to provide an approach which is strongly invariant to probability distribution, based upon comparisons between the received sample and a reference sample drawn from a noise-only population. Another example of an attempt to deal with the *a priori* problem is a paper by Schwartz, Harris and Hauptschein [39]. In this paper, the authors introduce Carnap's concept of inductive probability as a means of estimating the reliability of a channel by combining *a priori* knowledge with the evidence obtained from transmission. An example of the application of this method to establish the null zones in a decision feedback system is cited by the authors.

The theory of games has been used as a model of the radar jamming problem by Nilsson [40]. If the transmitter and jammer are constrained to a certain average power, the problem of selecting the spectral densities of the transmitted and jamming signals can be treated as a two-person zero-sum game.

Miscellaneous

In this section, we shall discuss several topics which have not received enough attention in the past three years to merit a special section in this report. The theory of sequential detection of signals is one area which seems likely to receive a good deal more attention in the future

than it has in the past. Blasblag [41] has formulated the problem of detecting signals in noise by quantizing a given random variable into two levels and using Bernoulli sequential detection. In another paper, Blasblag [42] applies Wald's sequential probability ratio test to the detection of a sine wave of arbitrary duty ratio in Gaussian noise, and in still another paper [43] some experimental results in sequential detection are presented.

The nonoptimum detection of distributed targets was treated in a paper by Stewart and Westerfield [44]. In this paper, the authors consider both reverberation and resolution problems in the detection of sonar signals. The loss of signal detectability caused by a hard limiter followed by a band-pass filter was investigated by Manasse, Price and Lerner [45]. The case of soft limiting was investigated by Galejs [46] who used the error function as a model of a smooth limiter. Galejs was able to calculate the SNR at the output of such a device followed by a narrow-band filter.

One simple alternative to making a binary decision at the receiver is to make a ternary decision by using, instead of one decision threshold, a pair of thresholds. Received signals falling in the null zone between thresholds result in no decision, and, unless later corrected, a "blank" appears in the output sequence. The improvement in the allowable information rate possible by the use of a null-zone reception was demonstrated by Bloom, *et al.* [47]. They show, for example, that under certain conditions the introduction of a single null zone achieves about half the improvement in information rate theoretically attainable by increasing the number of receiver levels without limit.

Harris, *et al.* [48], have investigated a number of cases in which the transmitter is notified of each occurrence of a "blank" and is asked for a repeat via a feedback channel that may or may not be error free. They have investigated the effect of terminating the process after a certain number of repeats, of various choices of permanent adjustments of the two thresholds, and of having time-varying threshold adjustments. A Gaussián distribution of the predecision noise was assumed. They also investigated the case where the predecision signal-to-Gaussian-noise ratio varies slowly as a function of time, requiring a continuous readjustment of threshold levels [49]. Cascades of such systems have also been treated [50]. Elias [51] described a method of supplementing a wide-band Gaussian noise channel with a similar analog channel in the reverse direction. By splitting each of these into subchannels, and by appropriate interconnections of the subchannels at both ends of the system, it is possible to reduce the complexity of coding required for forward transmission.

An interesting question for detection by discrete data processing was raised by Middleton [52]. It is almost universal practice, in such detection problems, to sample periodically, but Middleton asks whether it is possible that random sampling offers any advantage over conventional periodic sampling. He is able to show that in a wide variety of cases, periodic sampling is better and on this evidence, Middleton conjectures that periodic sampling is always better.

The solution of detection problems, when we are given data in a continuous closed interval, is often accomplished by taking a limiting form of some discrete problem. The validity of this approach was questioned in a paper by Slepian [53]. Slepian shows that, under certain conditions often assumed in detection studies, the detection of a Gaussian signal in Gaussian noise can be accomplished with arbitrarily small error. Furthermore, this detection may be based on a sample of received signal of arbitrarily short duration. Work in the next three years will undoubtedly shed more light on the problem of singular detection.

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PART 5: PREDICTION AND FILTERING

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MUCH OF THE research on prediction and filtering conducted in the United States during the period 1957–1960 was concerned essentially with various extensions of Wiener's theory. In particular, extensions involving nonstationary continuous-time processes, vector-valued processes, stationary and nonstationary discrete-time processes, non-Gaussian processes, incompletely specified processes, and nonlinear filters and predictors have received attention.

A new and very promising direction in prediction theory has been opened by the application of Bellman's dynamic programming to the determination of optimal adaptive filters and predictors. Actually, the basic work of Bellman and Kalaba [1]–[3], and its extensions and applications by Freimer [4], Aoki [5], Kalman and Koepcke [6], and Merriam [7] are not concerned with prediction and filtering as such. However, the recent work of Kalman [41] shows that, mathematically, there is a duality between the filtering problem and the control problems considered by Bellman and Kalaba, and others. Thus, these contributions are likely to have a considerable impact on the course of the development of the theory of filtering and prediction in the years ahead; they point toward an increasing utilization of digital computers and the concepts and techniques of discrete-state systems both in the design of predicting and filtering schemes and in their implementation.

During the past two years, four books containing in aggregate a substantial amount of material on prediction and filtering have been published. Davenport and Root [8] present a clear exposition of Wiener's theory and some of its extensions. Wiener's [9] monograph discusses orthogonal expansions of nonlinear functionals, but stops short

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of applying them to prediction problems. Bendat [10] presents a general survey of linear prediction and treats some special problems in considerable detail. Middleton's [11] weighty treatise contains a thorough exposition of the classical prediction theory together with a theory of reception in which the problems of prediction and filtering are formulated in the framework of decision theory. The appendix of Middleton's book includes an informative section on the solution of the Wiener-Hopf equation and some of its variants.

A more detailed discussion of the contributions to filtering and prediction theory is presented in the following pages. For convenience, the subjects of nonlinear filtering, nonstationary and discrete-time filtering, and miscellaneous contributions are dealt with separately.

Nonlinear Filtering

The contributions to nonlinear filtering and prediction have centered largely on the fundamental work of Wiener [12] and its earlier extensions by Bose [13] and Barrett [14]. A discernible trend in the research in this area is to consider special types of processes for which optimal nonlinear filters assume a simple form. A key work in this connection is that of Barrett and Lampard [15] in which the class, Λ,[1] of all second-order density functions admitting a diagonal representation of the form

$$p(x_1, x_2; \tau) = p(x_1)p(x_2) \sum_{n=0}^{\infty} a_n(\tau)\theta_n(x_1)\theta_n(x_2) \qquad (1)$$

is introduced. Here $p(x_1, x_2; \tau)$ denotes the second-order density of a stationary process $\{x(t)\}$, $x_1 = x(t)$, $x_2 = x(t + \tau)$, $p(x)$ is the first-order density, and $\{\theta_n(x)\}$ is a family of polynomials with the orthogonality property

$$\int p(x)\theta_m(x)\theta_n(x)\, dx = \delta_{mn}. \qquad (2)$$

In particular, Barrett and Lampard have shown that Gaussian and Rayleigh processes are of this type, with the θ_n being Hermite and Laguerre polynomials, respectively. Convergence and other aspects of the Barrett-Lampard expansion were investigated by Leipnik [16], while necessary and sufficient conditions under which $p(x_1, x_2; \tau)$ can be expressed in the form (1) have been given by J. L. Brown [17]. Brown also studied [18] a more general class of densities for which the expansion (1) is nondiagonal and the coefficients $a_{mn}(\tau)$ are restricted by the relation $a_{m1}(\tau) = d_m a_{11}(\tau)$, $m = 1, 2, \cdots$, the d_m being real constants. As shown by Brown, processes with densities of this type exhibit a number of interesting properties.

One way in which the Barrett-Lampard expansion can be used in nonlinear filtering was pointed out by Zadeh [19]. Specifically, assume that the second-order densit of a process with zero mean can be represented by (1 with the $\theta_n(x)$ not necessarily having the form of poly nomials. Then, if an optimal (minimum variance) filter i sought in the class of filters admitting the representatio

$$F(x) = \sum_{n=0}^{\infty} \int_0^{\infty} K_n(\tau)\theta_n[x(t - \tau)]\, d\tau, \qquad (3$$

where the $K_n(\tau)$ are undetermined kernels, and the desire output is written as

$$F^*(x) = \sum_{m \varepsilon M} \int_{-\infty}^{\infty} K_n^*(\tau)\theta_n[x(t - \tau)]\, d\tau, \qquad (4$$

where M is a finite index set and the $K_n^*(\tau)$ are give kernels, the determination of the $K_n(\tau)$ reduces to th solution of a finite number of Wiener-Hopf integra equations

$$\int_0^{\infty} K_m(\tau)a_m(t - \tau)\, d\tau = \int_{-\infty}^{\infty} K^*(\tau)a_m(t - \tau)\, d\tau, \qquad m \varepsilon M \qquad (5$$

with $K_n \equiv 0$, if $n \not\varepsilon M$.

Another type of process—for which the problem o determining an optimal nonlinear predictor is greatl simplified—was introduced by Nuttall [20]. Specifically Nuttall calls a process *separable*[2] if the conditional mea of x_2 given x_1 can be represented as

$$E\{x_2 \mid x_1\} = \int (x_2 - \mu)p(x_2; \tau \mid x_1)\, dx_2 = (x_1 - \mu)\rho(\tau), \qquad (6$$

where μ is the mean value of the process and $\rho(\tau)$ is it normalized autocorrelation function. Separable processe form a slightly broader class than that defined by Brow [18].

Among the many interesting properties of separabl processes is the following prediction property. Let $s(t$ be a signal mixed with additive noise. Then, if $\{s(t)\}$ is separable process, the best estimate of $s(t + \tau)$ in term of the best estimate of $s(t)$ is given by

$$s^*(t + \tau) = s^*(t)\rho_s(\tau) + \mu_s[1 - \rho_s(\tau)], \qquad (7$$

where $\rho_s(\tau)$ and μ_s are the normalized autocorrelation an the mean value of the signal process, and starred quantitie represent optimal (minimum variance) estimates. In th absence of noise, the explicit formula for the best pre dictor in terms of $s(t)$ becomes

$$s^*(t + \tau) = s(t)\rho_s(\tau) + \mu_s[1 - \rho_s(\tau)]. \qquad (8$$

Still another type of process for which the predictio problem is manageable was considered by D. A. Georg [21]. Here the observed signal $f(t)$ is assumed to be th

[1] In Barrett and Lampard's definition of Λ, $p(x_1, x_2; \tau)$ is not assumed to be symmetrical.

[2] It should be noted that the term "separable process" is used i the theory of stochastic processes in an altogether different sense

put of an invertible nonlinear system N preceded by invertible linear system L to which a white Gaussian nal $x(t)$ is applied. Thus, symbolically, $f = NLx$ and $= L^{-1}N^{-1}f$. Then, if an optimal estimate of $f(t + \alpha)$ enoted by $\bar{f}(t + \alpha)$, it is not difficult to find an operator acting on the present and past values of $x(t)$ such that $+ \alpha) = H_\alpha[x(t)]$. Once H_α has been found, $\bar{f}(t + \alpha)$ be expressed in terms of the present and past values (t) by the relation $\bar{f}(t + \alpha) = H_\alpha L^{-1}N^{-1}f$.

While some authors have sought to simplify the pretion problem by considering processes with special perties, others have turned to special types of nonear operators. In particular, the work of Bose [13], [22] s extended by D. A. Chesler [23] to operators of the m $F(\sum_{n=1}^{N} c_n\phi_n)$, where F denotes either a linear operator h memory, or a nonlinear memoryless operator, or a re general nonlinear operator possessing an inverse; the re adjustable constants, and the ϕ_n are nonlinear operas such that the expectation $E\{\phi_n(x)\phi_m(x)\} = 0$ for $\neq n$, x being the input to the filter. As was shown by se, in the absence of F the optimal value of each c can determined by measuring the mean-square error as a ction of, say, c_i and assigning to c_i the value which nimizes the mean-square error. This method is shown Chesler to be applicable also when F is a linear operator a nonlinear operator with no memory, The extension less straightforward when the only assumption on F hat it possesses a realizable inverse.

n all the foregoing analyses the signal process is umed to be stationary. However, there are many uations of practical interest in which an appropriate resentation for the signal is a series of the form

$$s(t) = \sum_{i=1}^{n} a_i\varphi_i(t), \tag{9}$$

which the $\varphi_i(t)$ are known functions of time and the are unknown constants or random variables. In such es, the problem of filtering or predicting $s(t)$ reduces the estimation of the coefficients a_i.

t was shown some time ago by Laning [24], that when the noise is additive, stationary, and Gaussian, 2) the nt distribution of the a_i is known, and 3) the loss ction $L(\epsilon)$ is non-negative and vanishes for $\epsilon = 0$, imal estimators for the a_i are memoryless nonlinear ctions of linear combinations of values of the input r the interval of observation. In a recent paper, ilar results were obtained by a different and more orous method by Kallianpur [25]. More specifically, for case where the interval of observation is $[0, T]$, and loss function is quadratic, Kallianpur derived explicit ressions for the best estimate of $s(t)$ at time $T + T_1$ erms of n linear functionals of the form $\int_0^T x(t)p_i(t)\,dt$, $= 1, 2, \cdots, n$, where $x(t)$ is the sum of signal and noise, l the $p_i(t)$ are square integrable solutions of integral ations

$$R(t - \tau)p_i(\tau)\,d\tau = \varphi_i(t), \qquad i = 1, 2, \cdots, n, \tag{10}$$

in which $R(\tau)$ is the correlation function of the process.

More concrete results for the same general problem were obtained by Middleton [26], and Glaser and Park [27]. In particular, Middleton found explicit expressions for minimum variance estimators of the a_i for the cases where 1) the a_i are jointly normally distributed, 2) the a_i are independent and Rayleigh distributed, 3) the a_i are independent and their distributions are not symmetrical, and 4) the a_i are independent and their distributions are symmetrical. Of these cases, only 1) and 4) yield linear estimators for the a_i.

The relation between maximum likelihood, minimum variance, and least squares estimates of the a_i was studied in earlier papers by Mann [28], and Mann and Moranda [29]. A number of interesting properties of minimum variance estimates of $s(t)$ and its derivatives for the case where the $\varphi_i(t)$ are polynomials in t were found by I. Kanter [30], [31]. A central result of Kanter is that an optimal weighting function for predicting the jth derivative of nth-degree polynomial can be expressed uniquely and simply in terms of optimal estimators of kth derivatives of kth-degree polynomials, with k ranging between j and n.

Filtering and Prediction of Nonstationary, Discrete-Time, and Mixed Processes

As is well-known [32], extensions of Wiener's theory to nonstationary processes lead to integral equations of the general form

$$\int_a^b R(t, \tau)x(\tau)\,d\tau = g(t), \qquad a \leq t \leq b, \tag{11}$$

in which $R(t, \tau)$ is the covariance function of the observed process. Little can be done toward the solution of this equation when $R(t, \tau)$ is an arbitrary covariance function. Thus, contributions to the theory of prediction of nonstationary continuous time processes consist essentially of methods of solving (11) in special cases.

Along these lines, Shinbrot [33] discussed the solution of (11) for the case where $R(t, \tau)$ can be expressed in the form

$$R(t, \tau) = \sum_{n=1}^{N} a_n(\tau)b_n(t),\ t > \tau \tag{12}$$

Using Shinbrot's methods, the solution of (11) reduces to the solution of a system of differential equations with time-varying coefficients. These is some advantage in such a reduction when one has available a differential analyzer or an equivalent machine. Similar results are yielded by a theory due to Darlington [34], [35], in which many of the concepts and techniques of time-invariant networks are extended to time-varying networks. As in the paper of Miller and Zadeh [32], a key assumption in these approaches is that the observed process may be generated by acting on white noise with a product of differential and inverse-differential operators, or equivalently, with a lumped-parameter linear time-varying

network. Darlington's paper [34] also contains a simplified technique for finding a finite memory Wiener filter for stationary signal and noise.

A special case for which explicit solution can be found has been studied by Bendat [36]. Here the basic assumption is that the signal is of the form $s(t) = 0$ for $t < 0$, $s(t) = \sum_1^N (a_n \cos nwt + b_n \sin nwt)$ for $t \geq 0$, where the a_n and b_m are random variables with known covariance matrices, while the covariance function of the noise is of the form

$$\begin{aligned} R(t_1, t_2) &= Ae^{-\beta|t_1-t_2|} \cos \gamma(t_1 - t_2) \quad \text{for} \quad t_1, t_2 \geq 0 \\ &= 0 \quad \text{for} \quad t_1 < 0 \quad \text{or} \quad t_2 < 0. \end{aligned} \tag{13}$$

Closely related cases in which the prediction problem can be solved completely are those in which the nonstationarity of signal and noise processes is due to a truncation (*e.g.*, multiplying the signal and noise by a step function) of stationary processes. This is also true in the case of discrete-time processes, as is demonstrated by several examples in Friedland's [37] extension of Wiener's theory to nonstationary sampled-data processes.

Several interesting results concerning the linear prediction of filtering of stationary discrete-time processes were described by Blum [38]–[40]. In particular, Blum has developed recursive formulas which express the estimate at time n in terms of a finite number of past estimates and past values of the observed process. This type of representation is especially useful in connection with so-called growing memory filters, *i.e.*, filters which act on the entire past of the input. Thus, if the input sequence (starting at $t = 0$) is denoted by $x_0, x_1, \cdots, x_n$, and the filter output at time n is denoted by z_n, then z_n is expressible as $z_n = \sum_{r=1}^n c_r x_r$, in which the c_r depend on n. A shortcoming of this representation is that as time advances the c_r have to be recomputed at each step and their number grows with n. On the other hand, a recursive relation (if it exists) is of the form

$$\begin{aligned} z_n = a_1 z_{n-1} + \cdots + a_k z_{n-k} \\ + b_0 x_n + b_1 x_{n-1} + \cdots + b_e x_{n-e}, \end{aligned} \tag{14}$$

where the a's, b's, k and e are constants independent of n, and hence, need not be recomputed. One complication in this approach to the problem is that in order to start the recursion one must know initially $z_0, z_1, \cdots, z_k$.

A somewhat related but more general approach has been formulated recently by Kalman [41]. Specifically, Kalman assumes that the observed process is an n-dimensional vector process $\{\mathbf{y}(t)\}$ which is generated by acting with a linear discrete-time system on a white noise $\{u(t)\}$; thus,

$$\begin{aligned} \mathbf{y}(t) &= P(t)\mathbf{x}(t) \\ \mathbf{x}(t+1) &= G(t)\mathbf{x}(t) + \mathbf{u}(t), \end{aligned} \tag{15}$$

where $\mathbf{X}(t)$ and $\mathbf{y}(t)$ are vectors and $P(t)$ and $G(t)$ are given time-varying matrices. (This assumption is analogous to the usual one in the case of nonstationary continuous-time prediction, *viz.*, that the observed process can be generated by acting on white noise with a time-varying network.) Kalman shows that an optimal (minimum variance) estimate of $x(t)$ is given by the recursive relation

$$\mathbf{x}^*(t+1) = [G(t) - A(t)P(t)]\mathbf{x}^*(t) + A(t)\mathbf{y}(t) \tag{16}$$

where

$$A(t) = G(t)M(t)P'(t)[P(t)M(t)P'(t)]^{-1}, \tag{17}$$

and $M(t)$ is given by

$$M(t+1) = [G(t) - A(t)P(t)]M(t)G'(t) + Q(t), \tag{18}$$

where G' is the transpose of G and $Q(t)$ is the covariance matrix $Q(t) = E\{\mathbf{u}(t)u'(t)\}$. The matrix $M(t)$ is the expectation of the matrix $\boldsymbol{\varepsilon}(t)\boldsymbol{\varepsilon}'(t)$, where $\boldsymbol{\varepsilon}(t)$ is the error at time t. In this formulation, to start the recursion one must know $x^*(0)$ and $M(0)$. However, in most cases the effect of the initial choices of $x^*(0)$ and $M(0)$ will be insignificant by the time the system reaches its steady state.

An interesting observation made by Kalman is that the prediction problem, in his formulation, is dual to a problem in control theory in which the objective is to find an input which minimizes a quadratic loss function.

In additions to extensions of Wiener's theory to nonstationary continuous- and discrete-time processes, extension to processes of mixed type were also reported. In particular, Robbins [42] solved the mean-square optimization problem for the case where the filter consists of a linear time-invariant system followed by a sampler which is followed in turn by another linear time-invariant system. Janos [43] gave a complete analysis of the case where a stationary signal is multiplied by a train of rectangular pulses, yielding a periodic pulse-modulated time series. The filter is assumed to be a time-invariant linear network. The integral equation satisfied by the impulsive response of the optimum filter is of the Wiener-Hopf type, but a multiplying factor involving trains of rectangular pulses complicates its solution. A method of solution of this equation is given by Janos for the infinite memory as well as the finite memory case.

Miscellaneous Contributions

There are several not necessarily unimportant problems in filtering and prediction which have received relatively little attention during the period under review. Contributions concerned with such problems are discussed briefly in this section.

It has long been recognized that the use of a quadratic loss function imposes a serious limitation on the applicability of Wiener's theory. Under certain conditions, however, optimality under the mean-square-error criterion implies optimality under a wide class of criteria. Such conditions have been found by Benedict and Sondhi [44

, independently, by Sherman [45]. Thus, Benedict Sondhi have shown that in the case of a Gaussian cess optimality with respect to a loss function of the n $L = \epsilon^2$, where ϵ denotes the error, implies optimality h respect to any loss function of the form $L = \sum_n |\epsilon|^n$, ere $n > 0$ but is not restricted to integral values. In rman's result, $L = f(\epsilon)$ is an even function and $\geq \epsilon_1 \geq 0$ implies $f(\epsilon_2) \geq f(\epsilon_1)$. More special cases olving the design of optimal filters under non-mean-are error criteria have been considered by Bergen [46] l Wernikoff [47]. A time-weighted mean-square-error erion which can be used to reduce the settling time of optimal linear filter was employed by Ule [48].

An extension of Wiener's theory to random parameter tems was described by Beutler [45]. In Beutler's mulation, the signal and noise are assumed to have ssed through a time-invariant random linear system ore being available for application to a filter or pre-tor. The linear system is assumed to be characterized a transfer function $H(w, \gamma)$, in which γ is a random rameter with a known distribution. In effect, this ounts to modifying the statistical characteristics of the ginal signal and noise processes.

The multiple series prediction problem for the infinite mory case was considered by Hsieh and Leondes [50]. their paper Hsieh and Leondes describe a simplified thod of solving the simultaneous integral equations for weighting functions. Their technique is not applicable, wever, to the finite memory case.

The optimization of continuous-time filters and pre-tors is frequently carried out by discretizing time and en letting the interval between successive samples proach zero. There are many published papers in ich limiting processes of this type are used without equate justification. A careful and rigorous analysis the problems involved in obtaining optimum con-uous-time linear estimates as limits of discrete-time imates was given by Swerling [51].

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On the Approach of a Filtered Pulse Train to a Stationary Gaussian Process*

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Summary—A narrow-band process is conveniently characterized in terms of a complex envelope whose magnitude is the envelope, and whose angle is the phase variation of the actual narrow-band process. When the narrow-band process is normally distributed, the complex envelope has the properties of a complex normally distributed process. This paper investigates the approach to the complex normally distributed form of the complex envelope of the output of a narrow-band filter when the input is wide-band non-Gaussian noise of a certain class, and the bandwidth of the narrow-band filter approaches zero. The non-Gaussian input consists of a train of pulses having identical waveshapes, but random amplitudes and phases. While the derivations assume statistical independence between pulses, it is shown that the results are valid for a certain interesting class of dependent pulses. The Central Limit Theorem is proved in the multidimensional case for the output process.

I. Introduction

There exist many situations in radar and communications problems in which the Central Limit Theorem is invoked to support the assumption that the output of a narrow-band filter with a wide-band non-Gaussian input possesses Gaussian statistics. However, little analytical work appears to have been done toward justifying this Gaussian assumption. This paper investigates the approach to stationary Gaussian statistics of the output of a narrow-band filter whose input is a sequence of pulses of random amplitude and phase. It is assumed that the pulses are of identical shape and occur periodically in time at a rate f_1 per second.

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II. The Narrow-Band Gaussian Process

A narrow-band process $N(t)$ centered at f_0 cps is representable in the form

$$N(t) = \mathrm{Re}\,\{\nu(t)e^{i2\pi f_0 t}\}, \quad (1)$$

where a property of $\nu(t)$, defined here as the *complex envelope* of $N(t)$, is that its magnitude is the conventional envelope of $N(t)$, while its angle is the conventional phase variation of $N(t)$ about the carrier phase $\omega_0 t$. The notation *Re* $\{x\}$ denotes the real part of x.

When $N(t)$ is a stationary Gaussian process, it is readily demonstrated that $\nu(t)$ has the properties of a stationary complex normally distributed process. The properties of a complex normal process are discussed by Doob[1] and Arens.[2] Arens deals with the *pre-envelope* of

[1] J. L. Doob, "Stochastic Processes," John Wiley and Sons, Inc., New York, N. Y., pp. 71–78; 1953.
[2] R. Arens, "Complex processes for envelopes of normal noise," IRE Trans. on Information Theory, vol. IT-3, pp. 204–207; September, 1957.

), which corresponds to $\nu(t)e^{j2\pi f_0 t}$ in our case. Be-se we are here specifically interested in narrow-band cesses, it is convenient to emphasize this by dealing h the complex envelope rather than with the pre-elope of $N(t)$. The typical jointly normal probability sity function and characteristic function for complex iates are given by Arens.[3] For our purposes, it will y be necessary to present the characteristic function he N jointly stationary random variables $\nu(t_j)$; $j = 1, \cdots N$,

$$F(\lambda_1, \cdots \lambda_N) = \exp\left[-\frac{1}{4}\sum_{p,q=1}^{N} R_\nu(t_p - t_q)\lambda_p\lambda_q^*\right], \qquad (2)$$

ere the λ's are the characteristic function variables, asterisk indicates the complex conjugate, and

$$R_\nu(\tau) = E[\nu^*(t)\nu(t+\tau)] = R_\nu^*(-\tau) \qquad (3)$$

defined[4] as the autocorrelation function of $\nu(t)$. As-ming that $\nu(t)$ has zero mean value, it must (according the definition of a complex normal variate) satisfy condition

$$E[\nu(t)\nu(t+\tau)] = 0. \qquad (4)$$

In the following sections, we will examine the complex velope $z(t)$ of a filtered random pulse train to determine conditions under which the $z(t)$ process may be said have properties approaching those indicated in (2)–(4).

III. Representation of Output Signal

In the subsequent discussion, we will deal entirely th complex time functions (complex envelopes and e-envelopes) because of the resulting simplification in rivations. It is convenient to conceive of the (complex) lse train as being generated at the output of a "pulse" ter by using a random complex area impulse train (t) as input, where

$$i_r(t) = \sum_{k=-\infty}^{\infty} \gamma_k\, \delta(t - kT_1), \qquad (5)$$

$t)$ is a unit impulse at $t = 0$, $T_1 = 1/f_1$, and γ_k is a ndom complex variable. While the subsequent analysis ll assume that γ_k is independent of γ_j, $j \neq k$, and entically distributed, the results of the analysis are tually applicable to a practically meaningful class of pendent γ's. Specifically, the analysis is valid for the pendent case if γ_j may be expressed as the output of a ime invariant) discrete filter whose input sequence tisfies the independence requirement. This fact is seen ost clearly in Fig. 1 where the output pre-envelope $)e^{i\omega_0 t}$ is shown as being obtained by three successive tering operations of $i_r(t)$. The first filter has an impulse sponse $i(t)\, d(t)$ where $i(t)$ is the unit impulse train

$$i(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_1), \qquad (6)$$

[3] *Ibid.*, see (11) and footnote 9.

[4] $E[\ ]$ will be used to denote an ensemble average.

and $d(t)$ is the pre-envelope of a continuous filter. Thus, the output of the first filter is a random complex-area impulse train with dependent complex areas. The second filter has an impulse response $P(t)$ which is one half[5] the pre-envelope of a typical normalized pulse. Then the output of the second filter is a periodic train of pulses of identical shape but having random amplitude and phase. The last filter is a narrow-band filter whose impulse response $\mu_1(t)e^{j2\pi f_0 t}$ is one half the pre-envelope of the physical narrow-band filter. Since this filter is "centered"[6] at f_0, $\mu_1(t)$ is one half the complex envelope of the filter impulse response.

Since the complex envelope of a narrow-band process centered at f_0 cps may be found by multiplication of the pre-envelope by $e^{-j2\pi f_0 t}$, and since this constitutes a spectrum shift of f_0 cps toward the origin, one may quickly verify that the output complex envelope $z(t)$ may be obtained as shown in Fig. 2.

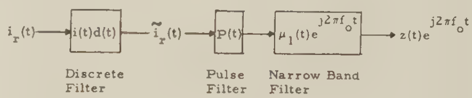

Fig. 1—Representation of output pre-envelope.

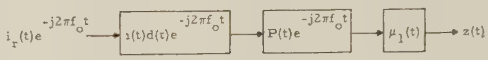

Fig. 2—Representation of output complex envelope.

For convenience in analysis, one may combine the three filters of Fig. 2 into one equivalent filter with impulse response $\mu(t)$, where

$$\mu(t) = [i(t)\, d(t)e^{-j2\pi f_0 t}] \otimes [P(t)e^{-j2\pi f_0 t}] \otimes \mu_1(t), \qquad (7)$$

and the symbol $\otimes$ denotes convolution. The conditions on convergence of $z(t)$ to a complex normally distributed process need then be stated only in terms of the equivalent filter.

In terms of $\mu(t)$, it may be seen that $z(t)$ is given by

$$z(t) = \sum_{k=-\infty}^{\infty} \gamma_k e^{-j2\pi f_0 k T_1}\mu(t - kT_1). \qquad (8)$$

It will be presumed at the outset that $\mu(t)$ is bounded—otherwise $z(t)$ will become infinite periodically.

IV. Average Requirements

Two requirements on $z(t)$ which are necessary for it to be a stationary complex normally distributed process are

$$R_z(\tau) = E[z^*(t)z(t+\tau)] = R_z^*(-\tau) \qquad (9)$$

and

$$E[z(t)z(t+\tau)] = 0. \qquad (10)$$

[5] The factor one half is needed since the pre-envelope of a filter output is one half the convolution of the pre-envelope of the input with the pre-envelope of the filter impulse response.

[6] "Centered" here means only that some convenient choice for f_0 has been made within the pass band of the filter.

These requirements will be called the *average requirements.* It is readily demonstrated that if, for a normally distributed process, the average requirements are not satified, many well-known properties of narrow-band Gaussian processes may not be obtained. For instance, the envelope may not be Rayleigh-distributed, or the phase may not be uniform over a 2π interval. This section is concerned with determining the conditions on the input impulse train and (equivalent) narrow-band filter leading to satisfaction by $z(t)$ of the average requirements. It should be noted that (assuming $E[|\ z(t)\ |^2]$ finite) (9) is just the requirement that $z(t)$ be a wide-sense stationary complex-valued random process.

Let the averages

$$\begin{aligned} E[\gamma_k \gamma_k^*] &= 1 \\ E[\gamma_k^2] &= \beta \end{aligned} \tag{11}$$

be defined. Then it is quickly determined that

$$E[z^*(t)z(t+\tau)] = \sum_{k=-\infty}^{\infty} \mu^*(t - kT_1)\mu(t + \tau - kT_1)$$

$$\equiv R_z(t, t+\tau). \tag{12}$$

By using $i(t)$, the unit impulse train [see (6)], one finds that

$$R_z(t, t+\tau) = i(t) \otimes \mu^*(t)\mu(t+\tau). \tag{13}$$

At this point, a frequency-domain interpretation becomes indispensable. Let the spectrum of $\mu^*(t)\mu(t+\tau)$ be defined as

$$X(f, \tau) = \int_{-\infty}^{\infty} \mu^*(t)\mu(t+\tau)e^{-i2\pi f t}\, dt. \tag{14}$$

It is interesting to note that this spectrum $X(f, \tau)$ is identical in form to Woodward's[7] ambiguity function for a radar pulse $\mu(t)$. Thus, it has several interesting properties. The reader is referred to the literature for these properties.[7–10]

The spectrum of the unit impulse train $i(t)$ is the frequency-domain impulse train given by

$$I(f) = \frac{1}{T_1} \sum_{m=-\infty}^{\infty} \delta(f - mf_1). \tag{15}$$

Thus, the spectrum of $R_z(t, t+\tau)$, defined as $P_z(f, \tau)$, is given by

$$P_z(f, \tau) = I(f)X(f, \tau)$$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{T_1} X(mf_1, \tau)\ \delta(f - mf_1). \tag{16}$$

Reverting back to the time domain,

$$R_z(t, t+\tau) = \frac{1}{T_1} X(0, \tau) + \frac{1}{T_1} e^{i2\pi f_1 t} X(f_1, \tau)$$

$$+ \frac{1}{T_1} e^{-i2\pi f_1 t} X(-f_1, \tau) + \cdots$$

$$= \frac{1}{T_1} \sum_{m=-\infty}^{\infty} X(mf_1, \tau) e^{i2\pi m f_1 t}. \tag{17}$$

Examination of (17) shows that $R_z(t, t+\tau)$ is periodic in t with a period T_1. In fact, (17) is just its Fourier series expansion. Thus, for $R_z(t, t+\tau)$ to be time independent, the fundamental and all harmonics must vanish; *i.e.*,

$$|\ X(mf_1, \tau)\ | = 0 \quad \text{for} \quad |\ m\ | > 0, \tag{18}$$

yielding

$$E[z^*(t)z(t+\tau)] = \frac{1}{T_1} \int_{-\infty}^{\infty} \mu^*(t)\mu(t+\tau)\, dt$$

$$\equiv R_z(\tau) = R_z^*(-\tau). \tag{19}$$

Let us assume that the spectrum of $\mu(t)$ is limited to a band B cps wide, centered at zero frequency. Then it is quickly determined that $X(f, \tau)$ cannot have a spectrum extending beyond a band $2B$ cps wide, centered at zero frequency. Let B_- and B_+ denote the lower and upper limit, respectively, of this band. Then it is quickly determined that (18) and thus, (19) will be satisfied if

$$\text{Max}\ \{|\ B_-\ |, B_+\} < f_1. \tag{20}$$

Fig. 3 shows $I(f)$ and $|\ X(f, \tau)\ |$ when (20) is satisfied.

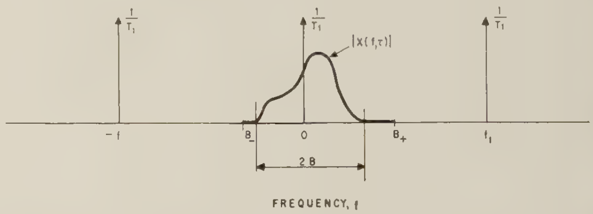

Fig. 3—Relation between filter pass band and pulse-train frequency for wide-sense stationary of $z(t)$.

Since a physical impulse response cannot be strictly band limited, it is clear that $R(t, t+\tau)$ can never be exactly time independent. However, it is also clear in the practical situation that its time dependence will be negligible if the "bandwidth" of $\mu(t)$ (defined in an appropriate sense) is sufficiently small.

Consider now the other average requirement,

$$E[z(t)z(t+\tau)]$$

$$= \beta \sum_{k=-\infty}^{\infty} e^{-i4\pi f_0 k T_1} \mu(t - kT_1)\mu(t + \tau - kT_1)$$

$$= \beta \tilde{R}_z(t, t+\tau). \tag{21}$$

Except for the special case $\beta = 0$, this expectation will not vanish unless the summation vanishes. This sum-

[7] P. M. Woodward, "Probability and Information Theory," McGraw-Hill Book Company, Inc., New York, N. Y.; 1953.

[8] R. M. Lerner, "Signals with uniform ambiguity functions," 1958 IRE National Convention Record, pt. 4, pp. 27–36.

[9] W. M. Siebert, "A radar detection philosophy," IRE Trans. on Information Theory, vol. IT–2, pp. 204–221; September, 1956.

[10] W. M. Siebert, "Studies of Woodward Uncertainty Function," Res. Lab. of Electronics, Mass. Inst. Tech., Cambridge, Quart. Prog. Rept.; April 15, 1958.

tion may be represented in the time domain as

$$\tilde{R}_z(t, t+\tau) = i(t)e^{-j4\pi f_0 t} \otimes \mu(t)\mu(t+\tau), \quad (22)$$

d in the frequency domain as

$$\tilde{P}_z(f, \tau) = I(f+2f_0)\tilde{X}(f, \tau), \quad (23)$$

ere $\tilde{P}_z(f, \tau)$ is the spectrum of $\tilde{R}_z(t, t+\tau)$, and $\tilde{X}(f, \tau)$ the spectrum of $\mu(t)\mu(t+\tau)$,

$$\tilde{X}(f, \tau) = \int_{-\infty}^{\infty} \mu(t)\mu(t+\tau)e^{-j2\pi f t}\, dt. \quad (24)$$

verting to the time domain,

$$\tilde{R}_z(t, t+\tau) = \frac{1}{T_1}\sum_{m=-\infty}^{\infty} \tilde{X}(mf_1 - 2f_0)e^{j2\pi(mf_1 - 2f_0)t}$$

$$= e^{-j4\pi f_0 t}\frac{1}{T_1}\sum_{m=-\infty}^{\infty} \tilde{X}(mf_1 - 2f_0)e^{j2\pi m f_1 t} \quad (25)$$

e function $\tilde{R}_z(t, t+\tau)$ is the product of the periodic nction $\exp(-j4\pi f_0 t)$ of period $1/2f_0$ by a summation ich is periodic with period T_1. The Fourier coefficients this sum are just $\tilde{X}(mf_1 - 2f_0)$. Thus, in order for $(t, t+\tau)$ to vanish (for satisfaction of the second erage requirement), it must be that

$$\tilde{X}(mf_1 - 2f_0) = 0 \quad \text{for all} \quad m. \quad (26)$$

$\mu(t)$ is band limited, then it may be shown that $\tilde{X}(f, \tau)$ nnot have a spectrum extending beyond the same nge as $X(f, \tau)$. Using the same definitions for B_- and B_+ for (20), it may readily be seen that (26) and thus 0) will be satisfied if the following two equations are tisfied:

$$B_+ < f_+$$
$$|B_-| < |f_-|, \quad (27)$$

here

$$f_+ = 2f_0 - Mf_1$$
$$f_- = 2f_0 - (M+1)f_1 = f_+ - f_1, \quad (28)$$

which

$$M = \operatorname*{Max}_m \{2f_0 - mf_1 > 0\}. \quad (29)$$

g. 4 shows $I(f - 2f_0)$ and $|\tilde{X}(f, \tau)|$ when (29) is satis- d. In the general case, *it should be noted that if either or f_- fall very close to zero frequency, then the second erage condition* (10) *will not be satisfied unless the filter ndwidth is very small.* If the transfer function of the uivalent filter is symmetrical and the gain drops to ro monotonically at the band edges it is clear that, far as making $\tilde{R}(t, t+\tau)$ small is concerned, a desirable ndition exists when $f_+ = |f_-| = f_1/2$. It may also be monstrated that when the filter transfer function is mmetrical, when $f_+ = |f_-| = f_1/2$, and when the γ_k's e real, *i.e.*, purely amplitude modulated pulses, then e real and imaginary parts of $z(t)$ become statistically dependent processes.

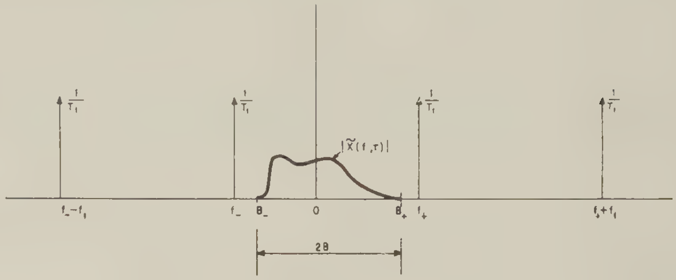

Fig. 4—Relation between filter pass band and pulse-train frequency for satisfaction of second average requirement.

To summarize this section briefly, simple frequency-domain inequalities (20) and (27) are presented which determine when the average requirements are satisfied by the complex envelope of the filtered pulse train for the case in which the equivalent narrow-band filter strictly band limits. When the narrow-band filter does not strictly band limit, the average requirements cannot be exactly satisfied. However, except in one singular case, they may be satisfied to any degree of precision if the "bandwidth" (suitably defined) of the narrow-band filter is sufficiently small. This singular case occurs when the center frequency of the narrow-band filter is an integral multiple of half the repetition frequency of the input pulse train. In this case, the second average requirement (10) can never be satisfied unless either

$$E[\gamma^2] = \beta \equiv 0,$$

or (30)

$$\int_{-\infty}^{\infty} \mu(t)\mu(t+\tau)\, dt \equiv 0$$

which are rather special situations.

In the non band-limited case, one may still define frequency limits B_+ and B_- beyond which the spectra $X(f, \tau)$ and $\tilde{X}(f, \tau)$ are negligibly small. Then inequalities of (20) and (27) are useful in determining whether the average requirements may be regarded as being satisfied.

V. Central Limit Theorem

In addition to satisfying the average requirements, the joint characteristic function of $z(t_1), z(t_2), \cdots z(t_N)$ must be of the form of (2) if $z(t)$ is to be a complex normally distributed stationary process. This section will demonstrate that the following two conditions are sufficient for the convergence of the joint characteristic function of $z(t_1), z(t_2), \cdots z(t_N)$ to the stationary complex normally distributed form [provided the second average requirement of (10) is satisfied and $\mu(t)$ is bounded]:

$$E[|\gamma|^3] < \infty,$$

$$\int_{-\infty}^{\infty} |\mu(t)|^2\, dt < \infty. \quad (31)$$

It will be convenient to standardize the complex variates $z_k \equiv z(t_k)$ to unit average-squared magnitude. Now

$$E[|z_k|^2] = \frac{1}{T_1}\int_{-\infty}^{\infty} |\mu(t)|^2\, dt \tag{32}$$

(assuming the first average requirement is satisfied). Thus, our standardization will be possible only if

$$\int |\mu(t)|^2\, dt$$

is finite. Assuming this to be the case, standardization of z_k will be effected by normalizing $\mu(t)$, so that

$$\frac{1}{T_1}\int_{-\infty}^{\infty} |\mu(t)|^2\, dt = 1. \tag{33}$$

The subsequent derivations can be made considerably more compact by dealing with suitably defined density functions and characteristic functions of multidimensional complex variates. The density function and characteristic function of a one-dimensional complex variate will now be defined and the generalization to N-dimensional complex vectors will be clear to the reader. Let v and u be the real and imaginary parts of a complex variate w given by

$$w = v + ju, \tag{34}$$

and let $P(v, u)$ be the joint density function of v and u. Then the density function of w, $P_1(w)$ will be defined as

$$P_1(w) = P[\mathrm{Re}\,\{w\}, \mathrm{Im}\,\{w\}]. \tag{35}$$

An average of some function of w, $g(w)$, with respect to $P_1(w)$ is to be interpreted as

$$\int g(w)P_1(w)\, dw = \iint g(u + jv)P(u, v)\, du\, dv. \tag{36}$$

If the joint characteristic function $F(\xi, \eta)$ of the variables u, v is defined as

$$F(\xi, \eta) = \iint P(u, v)e^{j(\xi u + \eta v)}\, du\, dv, \tag{37}$$

then the characteristic function of w is defined as

$$F_1(\lambda) = \int P_1(w)e^{jRe(w^*\lambda)}\, dw = F[\mathrm{Re}\,(\lambda), \mathrm{Im}\,(\lambda)], \tag{38}$$

where the complex characteristic function variable

$$\lambda = \xi + j\eta. \tag{39}$$

An overline will be used to indicate a vector, or a collection of variables. Thus, the symbol $\bar{w}$ may be used to denote the set of complex variables $(w_1, w_2, \cdots w_N)$. An average of some function of $\bar{w}$, $g(\bar{w})$, is to be interpreted as

$$\int g(\bar{w})P_1(\bar{w})\, d\bar{w}$$

$$= \iint \cdots \int g(u_1 + jv_1, u_2 + jv_2, \cdots u_N + jv_N)$$

$$\cdot P(u_1, v_1, u_2, v_2, \cdots u_N, v_N)\, du_1\, dv_1 \cdots du_N\, dv_N \tag{40}$$

where $P_1(\bar{w})$ is the density function of an N-dimensional complex variate $\bar{w}$, and $P(u_1, v_1, \cdots u_N, v_N)$ is the joint density function of the real and imaginary parts of the components of $\bar{w}$. The characteristic function of $\bar{w}$ is expressed as

$$F_1(\bar{\lambda}) = \int P_1(\bar{w})e^{jRe(\bar{w}^*\cdot\bar{\lambda})}\, d\bar{w}, \tag{41}$$

where $\bar{\lambda}$ is a complex vector $(\lambda_1, \lambda_2, \cdots \lambda_N)$, and $\bar{w}^* \cdot \bar{\lambda}$ is the inner or dot product of the vectors $\bar{w}^*$ and $\bar{\lambda}$.

The complex random variable z_k is given by

$$z_k = \sum_{m=-\infty}^{\infty} \gamma_m\mu(t_k - mT_1)e^{-j2\pi f_0 mT_1} = \sum_{m=-\infty}^{\infty} z_{mk}, \tag{42}$$

where the random variable

$$z_{mk} = \gamma_m\mu(t_k - mT_1)e^{-j2\pi f_0 mT_1}. \tag{43}$$

If the vector $\bar{z}$ denotes the set of N complex variates $(z_1, z_2, \cdots z_N)$, then it is representable as a sum of independent complex vectors as follows:

$$\bar{z} = \sum_{m=-\infty}^{\infty} \bar{z}_m, \tag{44}$$

where the vector $\bar{z}_m$ denotes the set of N complex variates $(z_{m1}, z_{m2}, \cdots z_{mN})$.

If the probability density function of the γ_k is denoted by W_γ, then the probability density function of $\bar{z}_m$, $W_m(\bar{z}_m)$, is readily demonstrated to be given by

$$W_m(\bar{z}_m) = \frac{1}{|C_{m1}|^2} W_\gamma\left[\frac{z_{m1}}{C_{m1}}\right] \prod_{n=2}^{N} \delta\left[z_{mn} - z_{m1}\frac{C_{mn}}{C_{m1}}\right], \tag{45}$$

where the coefficient

$$C_{mn} = \mu(t_n - mT_1)e^{-j2\pi f_0 mT_1}, \tag{46}$$

and $\delta(x)$ is a unit impulse at $x = 0$.

From (45) and (41), the characteristic function of $\bar{z}_m$ is found to be simply

$$F_m(\bar{\lambda}) = F_\gamma[\bar{C}_m^*\cdot\bar{\lambda}], \tag{47}$$

where F_γ is the common characteristic function of the γ_k, and the coefficient vector $\bar{C}_m$ denotes the set of coefficients $(C_{m1}, C_{m2}, \cdots C_{mN})$. The vector $\bar{\lambda}$ denotes the set of N complex characteristic function variables $(\lambda_1, \lambda_2, \cdots \lambda_N)$. Inasmuch as $\bar{z}$ is represented as a sum of independent random variables, its characteristic function $F(\bar{\lambda})$ is given by the product of the characteristic functions of the component random variables and the logarithm of $F(\bar{\lambda})$ by the sum

$$\log F(\bar{\lambda}) = \sum_{m=-\infty}^{\infty} \log F_\gamma[\bar{C}_m^*\cdot\bar{\lambda}]. \tag{48}$$

It is readily demonstrated that if

$$\beta_3 = E[|\gamma|^3] < \infty, \tag{49}$$

then $F_\gamma(\lambda)$ has the finite series expansion

$$F_\gamma(\lambda) = 1 - \tfrac{1}{4}[|\lambda^2| + \mathrm{Re}\,\{\lambda^2\beta^*\}] + \tfrac{1}{6}\vartheta\beta_3\,|\lambda|^3, \tag{50}$$

where ϑ is a complex quantity of modulus not exceeding unity. Thus,

$$\bar{\lambda}) = 1 - \tfrac{1}{4}[|\overline{C_m^*}\cdot\bar{\lambda}|^2 + \text{Re}\{\beta^*(\overline{C_m^*}\cdot\bar{\lambda})^2\}] + \tfrac{1}{6}\mathcal{I}\beta_3|\overline{C_m^*}\cdot\bar{\lambda}|^3. \quad (51)$$

We are interested in the behavior of $F_m(\bar{\lambda})$ as the filter ndwidth (defined in an appropriate sense) approaches o. To study this behavior, let $\mu(t)$ be expressed in the m

$$\mu(t) = \sqrt{B}\, s(Bt), \quad (52)$$

ere $s(t)$ has unit bandwidth and $\mu(t)$ has bandwidth B. change in bandwidth of $\mu(t)$ is then a scale change in time domain (and also in the frequency domain). e factor $\sqrt{B}$ is needed to maintain the normalization (33). With the aid of (52), (51) may be represented as

$$F_m(\bar{\lambda}) = 1 - Bg_1 + B^{3/2}g_2, \quad (53)$$

ere the functions

$$g_1 = \tfrac{1}{4}[|\overline{D_m^*}\cdot\bar{\lambda}|^2 + \text{Re}\{\beta^*(\overline{D_m^*}\cdot\bar{\lambda})^2\}], \quad g_2 = \tfrac{1}{6}\mathcal{I}\beta_3|\overline{D_m^*}\cdot\bar{\lambda}|^3. \quad (54)$$

e normalized coefficient vector D_m is given by

$$\overline{D_m} = \frac{1}{\sqrt{B}}\overline{C_m}. \quad (55)$$

te that as $B \to 0$, $\overline{D_m} \to \mu(0)e^{-i2\pi f_0 m T_1}\,\bar{U}$ where $\bar{U}$ is vector with unit values for coordinates. Thus for fixed both Bg_1 and $B^{3/2}\,g_2$ approach zero as B approaches o. It may then be shown that for *sufficiently small B*, e may represent $\log F_m(\bar{\lambda})$ by the finite series

$$\log F_m(\bar{\lambda}) = -\tfrac{1}{4}[|\overline{C_m^*}\cdot\bar{\lambda}|^2 + \text{Re}\{\beta^*(\overline{C_m^*}\cdot\bar{\lambda})^2\}] + \tfrac{1}{3}\mathcal{I}_1\beta_3|\overline{C_m^*}\cdot\bar{\lambda}|^3, \quad (56)$$

ere I_1 is a complex quantity of modulus not exceeding ity. It follows that for sufficiently small B,

$$\log F(\bar{\lambda}) = -\tfrac{1}{4}\sum_{m=-\infty}^{\infty}|\overline{C_m^*}\cdot\bar{\lambda}|^2 + \tfrac{1}{4}\text{Re}\left\{\beta^*\sum_{m=-\infty}^{\infty}(\overline{C_m^*}\cdot\bar{\lambda})^2\right\} + \frac{\beta_3}{3}\sum_{m=-\infty}^{\infty}\mathcal{I}_1|\overline{C_m^*}\cdot\bar{\lambda}|^3. \quad (57)$$

If the average requirements are met,

$$\sum_{-\infty}^{\infty}|\overline{C_m^*}\cdot\bar{\lambda}|^2 = \sum_{p,q=1}^{N}\lambda_p\lambda_q^*\sum_{m=-\infty}^{\infty}C_{mp}^*C_{mq}$$
$$= \sum_{p,q=1}^{N}R_z(t_p - t_q)\lambda_p\lambda_q^*\sum_{m=-\infty}^{\infty}(\overline{C_m^*}\cdot\bar{\lambda})^2$$
$$= \sum_{p,q=1}^{N}\lambda_p\lambda_q\sum_{m=-\infty}^{\infty}C_{mp}C_{mq}$$
$$= \sum_{p,q=1}^{N}\tilde{R}_z(t_p - t_q)\lambda_p\lambda_q = 0. \quad (58)$$

w the last sum in (57) is bounded as shown in (59):

$$\sum_{-\infty}^{\infty} I_1|\overline{C_m^*}\cdot\bar{\lambda}|^3 \leq \sum_{p,q,r=1}^{N}|\lambda_p\lambda_q\lambda_r|\sum_{m=-\infty}^{\infty}|C_{mp}C_{mq}C_{mr}|. \quad (59)$$

If we define

$$t_k = t - \tau_k, \quad (60)$$

then the summation over m becomes

$$\sum_{m=-\infty}^{\infty}|C_{mp}C_{mq}C_{mr}| = i(t)\otimes|\mu(t-\tau_p)\mu(t-\tau_q)(t-\tau_r)|, \quad (61)$$

where $i(t)$ is the unit impulse train of (6). By using a frequency domain interpretation, it is seen that if we assume that the spectrum of

$$|\mu(t-\tau_p)\mu(t-\tau_q)\mu(t-\tau_r)|$$

is confined to frequencies within the region $-f_1 < f < f_1$, then the above sum becomes

$$\sum_{m=-\infty}^{\infty}|C_{mp}C_{mq}C_{mr}| = \frac{1}{T_1}\int_{-\infty}^{\infty}|\mu(t-\tau_p)\mu(t-\tau_q)\mu(t-\tau_r)|\,dt. \quad (62)$$

However, it is readily shown by a simple application of the Hölder inequality for integrals[11] that

$$\int_{-\infty}^{\infty}\prod_{k=1}^{l}|\mu(t-\tau_{\nu k})|\,dt < \int_{-\infty}^{\infty}|\mu(t)|^l\,dt = B^{l/2-1}\int_{-\infty}^{\infty}|s(t)|^l\,dt. \quad (63)$$

Thus, the last sum in (57) is bounded by

$$\sqrt{B}\,\frac{1}{T_1}\int_{-\infty}^{\infty}|s(t)|^3\,dt\left\{\sum_{k=1}^{N}|\lambda_k|\right\}^3. \quad (64)$$

Since $s(t)$ is bounded and of integrable squared magnitude,

$$\int_{-\infty}^{\infty}|s(t)|^3\,dt$$

is finite. Thus for fixed $\bar{\lambda}$, the last sum approaches zero as $\sqrt{B}$. It follows that

$$\lim_{B\to 0}\log F(\bar{\lambda}) = -\tfrac{1}{4}\sum_{p,q=1}^{N}R_z(t_p - t_q)\lambda_p\lambda_q^*, \quad (65)$$

or equivalently

$$\lim_{B\to 0}F(\bar{\lambda}) = \exp\left\{-\tfrac{1}{4}\sum_{p,q=1}^{N}R_z(t_p - t_q)\lambda_p\lambda_q^*\right\}, \quad (66)$$

which is the normal characteristic function.

The continuity theorem for characteristic functions[12] may be applied to show that the joint distribution function of $z(t_1)$, $z(t_2)$, $\cdots$ $z(t_N)$ converges to the normal distribution function.

[11] G. H. Hardy, J. E. Littlewood, and G. Polya, "Inequalities," Cambridge University Press, Cambridge, Eng., p. 140; 1959.
[12] H. Cramer, "Mathematical Methods of Statistics," Princeton University Press, Princeton, N. J.; 1946.

The Axis Crossings of a Stationary Gaussian Markov Process*

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Summary—In a stationary Gaussian Markov process (or Ornstein-Uhlenbeck process) the expected number of axis crossings per unit time, the probability density of the lengths of axis-crossing intervals, and the probability of recurrence at zero level do not exist as ordinarily defined. In this paper new definitions are presented and some asymptotic formulas are derived. Certain renewal equations are approximately satisfied, thereby suggesting an asymptotic approach to independence of the lengths of successive axis-crossing intervals. Mention is made of an application to the filter-clip-filter problem.

Introduction

In two previous papers[1,2] the author has described a theory of the axis-crossing intervals of a stationary random process $\xi(t)$. Following the work of Rice,[3] relations were given between the following quantities:

1) β, the expected number of axis crossings per unit time;
2) $P_0(\tau)$, the probability density of the lengths of intervals between successive axis crossings;
3) $U(\tau)\,d\tau$, the probability of a crossing in $(t + \tau, t + \tau + d\tau)$, given a crossing in $(t - dt, t)$;
4) $r(\tau)$, the autocorrelation function of the given process after infinite clipping;
5) $w(\tau_1, \tau_2, \tau_3)$, the fourth product moment of the process after infinite clipping.

Suppose, however, that $\xi(t)$ is a stationary Gaussian Markov process, *i.e.*, an Ornstein-Uhlenbeck process, or the output of an RC low-pass filter when the input is stationary, white Gaussian noise. In this case the previous theory breaks down, since β, $P_0(\tau)$, and $U(\tau)$ do not exist as defined above. It is the purpose of this paper to extend the theory to include the stationary Gaussian Markov process and to examine some of the consequences.

Probability of One or More Crossings

As was shown by Rice,[4] the expected number of crossings per unit time is infinite for a stationary Gaussian Markov process. For this reason, the previous definition of β must be generalized.

Let $\beta(\Delta)\Delta$ be the probability that one or more crossings occur in the finite interval $(t - \Delta, t)$. [As $\Delta \to 0$, $\beta(\Delta)$ becomes the constant β, as previously defined, for processes in which the limit exists.]

Now suppose that $\xi(t)$ is a stationary Gaussian Markov process. The mean value of $\xi(t)$ is assumed to be zero. The autocorrelation function of $\xi(t)$ is exponential,[5] and it is convenient to set the time constant equal to unity. Thus the normalized autocorrelation function is as follows:

$$\rho(\tau) = e^{-|\tau|}. \tag{1}$$

For such a process, the probability $p(0, \tau)$ that a given interval of length τ contains *no* crossings is the function,

$$p(0, \tau) = \frac{2}{\pi} \sin^{-1} (e^{-\tau}). \tag{2}$$

This result was derived by Siegert[6] and by Slepian.[7]

Thus, for this process, the quantity $\beta(\Delta)\Delta$ is given by the relation

$$\beta(\Delta)\, \Delta = 1 - \frac{2}{\pi} \sin^{-1} (e^{-\Delta}). \tag{3}$$

If Δ is small [*i.e.*, compared to unity, the time constant in (1)], then asymptotically,

$$\beta(\Delta) = \frac{2\sqrt{2}}{\pi}\, \Delta^{-1/2} + 0(\Delta^{1/2}). \tag{4}$$

As was stated previously, $\beta(\Delta)$ does not remain finite as $\Delta \to 0$. The nature of the singularity is apparent from (4).

In a previous paper,[8] it was shown that β is proportional to the initial slope of $r(\tau)$, the autocorrelation function of $\xi(t)$ after infinite clipping. That derivation cannot be generalized under the present definition of $\beta(\Delta)$.

* Received by the PGIT, August 2, 1960. This work was supported by the Office of Nav. Res. under Contract Nonr-1100(15).

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[1] J. A. McFadden, "The axis-crossing intervals of random functions," IRE Trans. on Information Theory, vol. IT-2, pp. 146–150; December, 1956.

[2] J. A. McFadden, "The axis-crossing intervals of random functions II," IRE Trans. on Information Theory, vol. IT-4, pp. 14–24; March, 1958.

[3] S. O. Rice, "Mathematical analysis of random noise," *Bell Sys. Tech. J.*, vol. 23, pp. 282–332, July, 1944; vol. 24, pp. 46–156; January, 1945. See esp. sect. 3.3 and 3.4.

[4] Rice, *op. cit.*, sect. 3.3.

[5] J. L. Doob, "The Brownian movement and stochastic equations," *Ann. Math.*, vol. 43, pp. 351–369 (1.1.6); April, 1942.

[6] A. J. F. Siegert, "On the Roots of Markoffian Random Functions," RAND Corp., Santa Monica, Calif., Rept. No. RM-447; September, 1950.

[7] S. O. Rice, "Distribution of the duration of fades in radio transmission," *Bell Sys. Tech. J.*, vol. 37, pp. 581–635(114); May, 1958.

[8] McFadden, *op. cit.*, "The axis-crossing intervals of random functions," (12).

Probability Density of the Length of an Axis-Crossing Interval

As was shown by Kohlenberg[9] and by the author,[10] e probability density $P_0(\tau)$ of the length of an interval tween successive axis crossings in a stationary process ordinarily equal to $p''(0, \tau)/\beta$. It is not surprising that ch a formula fails in the Gaussian Markov case. The finition of $P_0(\tau)$ must therefore be revised.

Consider those sample functions $\xi(t)$ for which one or ore crossings have occurred in $(t - \Delta, t)$. Let T be random variable such that the next crossing after time t curs at time $t + T$. Then $P_\Delta(\tau)\, d\tau$ is defined as the obability that T lies in the range between τ and $\tau + d\tau$. f. the "horizontal window condition" of Kac and epian.[11] As $\Delta \to 0$, $P_\Delta(\tau)$ becomes the density $P_0(\tau)$, previously defined, for processes in which the limit ists.]

$P_\Delta(\tau)$ will now be expressed in terms of $\beta(\Delta)$ and $(0, \tau)$. Consider the following events E_1, E_2, and E_3:[12]

E_1: No crossings occur in the interval $(t, t + \tau)$.
E_2: No crossings occur in the interval $(t - \Delta, t + \tau)$.
E_3: One or more crossings occur in $(t - \Delta, t)$, but none in $(t, t + \tau)$.

Then the probabilities of these events are related as llows:

$$P\{E_1\} = P\{E_2\} + P\{E_3\}; \tag{5}$$

r

$$p(0, \tau) = p(0, \tau + \Delta) + \beta(\Delta)\, \Delta \int_\tau^\infty P_\Delta(l)\, dl. \tag{6}$$

he last term of (6) is equivalent to $P\{E_3\}$ because E_3 s the event that one or more crossings occur in $(t - \Delta, t)$ nd that the next crossing occurs after time $t + \tau$.

After differentiation with respect to τ, (6) yields the ollowing solution for $P_\Delta(\tau)$:

$$P_\Delta(\tau) = \frac{p'(0, \tau + \Delta) - p'(0, \tau)}{\beta(\Delta)\, \Delta}. \tag{7}$$

In those cases in which the limit exists, (7) becomes the revious expression $p''(0, \tau)/\beta$ as $\Delta \to 0$.]

For a stationary Gaussian Markov process, by (2) nd (3),

$$P_\Delta(\tau) = \frac{e^{-\tau}(1 - e^{-2\tau})^{-1/2} - e^{-(\tau+\Delta)}[1 - e^{-2(\tau+\Delta)}]^{-1/2}}{\frac{\pi}{2} - \sin^{-1}(e^{-\Delta})}. \tag{8}$$

As $\tau \to 0$, $P_\Delta(\tau)$ behaves like $\tau^{-1/2}$. An asymptotic formula for $P_\Delta(\tau)$ cannot easily be given (for small Δ) which remains valid as $\tau \to 0$.

The Laplace transform of $P_\Delta(\tau)$ is more manageable. By the use of tables,[13]

$$p_\Delta(s) \equiv \int_0^\infty e^{-s\tau} P_\Delta(\tau)\, d\tau$$

$$= \frac{B\left(\frac{s}{2} + \frac{1}{2}, \frac{1}{2}\right)}{\pi - 2\sin^{-1}(e^{-\Delta})} \left\{1 - e^{s\Delta} I_{(e^{-2\Delta})}\left(\frac{s}{2} + \frac{1}{2}, \frac{1}{2}\right)\right\}, \tag{9}$$

where

$$I_x(p, q) = \frac{\int_0^x \xi^{p-1}(1 - \xi)^{q-1}\, d\xi}{B(p, q)}. \tag{10}$$

The following asymptotic expansion for $p_\Delta(s)$ exists for small Δ:

$$p_\Delta(s) = 1 - \frac{sB\left(\frac{s}{2} + \frac{1}{2}, \frac{1}{2}\right)}{2\sqrt{2}} \Delta^{1/2} + \tfrac{2}{3} s\, \Delta + 0(\Delta^{3/2}). \tag{11}$$

As $\Delta \to 0$ this becomes the transform of a δ function.

Asymptotic expressions for the first and second moments of T follow from (11).

$$E(T) = \frac{\pi}{2\sqrt{2}} \Delta^{1/2} + 0(\Delta). \tag{12}$$

$$E(T^2) = \frac{\pi \log 2}{\sqrt{2}} \Delta^{1/2} + 0(\Delta^{3/2}). \tag{13}$$

The ratio of the variance $D^2(T)$ to the mean $E(T)$ has a finite limit, since

$$\frac{D^2(T)}{E(T)} = 2 \log 2 + 0(\Delta^{1/2}). \tag{14}$$

In the previous theory,[1,2] $E(T) = 1/\beta$. A similar relation holds here too, asymptotically, for by (4) and (12),

$$\beta(\Delta)E(T) = 1 + 0(\Delta^{1/2}). \tag{15}$$

Probability of Recurrence

For a general stationary Gaussian process, the probability $U(\tau)\, d\tau$ of a crossing in $(t + \tau, t + \tau + d\tau)$, given a crossing in $(t - dt, t)$, was derived by Rice.[14] If, however, $\xi(t)$ is a Markov process, then Rice's derivation is not applicable, since $\xi'(t)$ has an infinite variance.

In a previous paper by the author,[15] $U(\tau)$ was related to the fourth product moment after infinite clipping.

9 A. Kohlenberg, "Notes on the Zero Distribution of Gaussian Voise," M. I. T. Lincoln Lab., Lexington, Mass., Tech. Memo. 44, . 4; October, 1953.

10 McFadden, *op. cit.*, "The axis-crossing intervals of random unctions II," (4).

11 M. Kac and D. Slepian, "Large excursions of Gaussian processes," *Ann. Math. Stat.*, vol. 30, pp. 1215–1228; December, 1959.

12 McFadden, *op. cit.*, "The axis-crossing intervals of random unctions II," Appendix I.

13 Bateman Manuscript Project, "Tables of Integral Transforms," McGraw-Hill Book Co., Inc., New York, N. Y., vol. 1, p. 261(1); p. 129(4), (5); 1954.

14 Rice, *op. cit.*, (3.4–10).

15 McFadden, *op. cit.*, "The axis-crossing intervals of random functions II," Appendix II.

That derivation has been extended below. Let $U_{\Delta\delta}(\tau)\delta$ be the conditional probability that one or more crossings occur in $(t + \tau, t + \tau + \delta)$, given that one or more crossings have occurred in $(t - \Delta, t)$. Then $\beta(\Delta)U_{\Delta\delta}(\tau)\Delta\delta$ is the joint probability of one or more crossings in $(t - \Delta, t)$ and one or more crossings in $(t + \tau, t + \tau + \delta)$. [As $\Delta \to 0$ and $\delta \to dt$, $U_{\Delta\delta}(t)$ becomes $U(\tau)$, as previously defined, for processes in which the limit exists.]

The number of crossings in $(t - \Delta, t)$ and the number in $(t + \tau, t + \tau + \delta)$ can be either odd or even. The four events "odd-odd," "odd-even," "even-odd" and "even-even" have nearly the same probability. Because of the clustering of axis crossings in a stationary Gaussian Markov process, the difference between any two of these probabilities is of higher order in Δ or δ; in other words, only a minute change in the magnitude of δ or Δ is necessary to change the number of crossings by one. Thus $\beta(\Delta)U_{\Delta\delta}(\tau)\Delta\delta$ is equal to four times the probability that a net sign change occurs in $(t - \Delta, t)$ and another in $(t + \tau, \ + \tau + \delta)$, plus higher-order terms in Δ and δ.

Let P_{-++-} be the probability that $\xi(t - \Delta) < 0$, $\xi(t) \geq 0$, $\xi(t + \tau) \geq 0$, and $\xi(t + \tau + \delta) < 0$, and correspondingly for other combinations of signs. Then for small Δ and δ,

$$\beta(\Delta)U_{\Delta\delta}(\tau)\,\Delta\delta \sim 4(P_{-++-} + P_{-+-+} + P_{+--+} + P_{+-+-}), \tag{16}$$

or by symmetry,

$$\beta(\Delta)U_{\Delta\delta}(\tau)\,\Delta\delta \sim 8(P_{-++-} + P_{-+-+}). \tag{17}$$

In another paper by the author,[16] the various probabilities P_{-++-}, etc., have been expressed in terms of moments of the process $\xi(t)$ after infinite clipping. Let

$$\begin{aligned} x(t) &= +1 \quad \text{when} \quad \xi(t) \geq 0; \\ &= -1 \quad \text{when} \quad \xi(t) < 0. \end{aligned} \tag{18}$$

Let $x_1 = x(t - \Delta)$, $x_2 = x(t)$, $x_3 = x(t + \tau)$, and $x_4 = x(t + \tau + \delta)$. Furthermore, let the correlation coefficients after clipping be $r_{ij} = E(x_ix_j)$, and let the fourth product moment be $w = E(x_1x_2x_3x_4)$. Then[16]

$$\begin{aligned} P_{-++-} &= \frac{1}{16}\left[1 - r_{12} - r_{13} + r_{14} + r_{23} - r_{24} - r_{34} + w\right]; \\ P_{-+-+} &= \frac{1}{16}\left[1 - r_{12} + r_{13} - r_{14} - r_{23} + r_{24} - r_{34} + w\right]; \end{aligned} \tag{19}$$

and (17) becomes

$$\beta(\Delta)U_{\Delta\delta}(\tau)\,\Delta\delta \sim 1 - r_{12} - r_{34} + w. \tag{20}$$

[16] J. A. McFadden, "Urn models of correlation and a comparison with the multivariate normal integral," *Ann. Math. Stat.*, vol. 26, pp. 478–489; September, 1955. See sect. 6.

Since $\xi(t)$ is Gaussian, r_{ij} is given by the arcsine formula,[17]

$$r_{ij} = \frac{2}{\pi}\sin^{-1}\rho_{ij}, \tag{21}$$

where ρ_{ij} is the correlation coefficient before clipping. A series for the fourth product moment of a stationary Gaussian Markov process, after clipping, has been derived by McFadden.[18] The result is

$$\begin{aligned} w = \frac{4}{\pi^2}\Bigg\{&\sin^{-1}\rho_{12}\sin^{-1}\rho_{34} + \rho_{12}\rho_{34}(1 - \rho_{12}^2)^{1/2}(1 - \rho_{34}^2)^{1/2} \\ &\cdot \sum_{m=1}^{\infty} \frac{(-\frac{1}{2})_m(-\frac{1}{2})_m}{(\frac{1}{2})_m}\frac{\rho_{23}^{2m}}{m!}F(1 - m, 1; \tfrac{3}{2} - m; \rho_{12}^2) \\ &\cdot F(1 - m, 1; \tfrac{3}{2} - m; \rho_{34}^2)\Bigg\}, \end{aligned} \tag{22}$$

where $(a)_m = \Gamma(a + m)/\Gamma(a)$. In (21) and (22),

$$\rho_{12} = e^{-\Delta}; \qquad \rho_{23} = e^{-\tau}; \qquad \rho_{34} = e^{-\delta}. \tag{23}$$

Now for small Δ and δ, the arcsines may be expanded as in (4). Furthermore, since $(1 - \rho_{12}^2)^{1/2}$ is of $0(\Delta^{1/2})$, the arguments of the hypergeometric functions may be replaced by unity, the errors being of higher order. Thus for $m \geq 1$,[19]

$$\begin{aligned} (1 - \rho_{12}^2)^{1/2}&F(1 - m, 1; \tfrac{3}{2} - m; \rho_{12}^2) \\ &= (2\,\Delta)^{1/2}F(1 - m, 1; \tfrac{3}{2} - m; 1) + 0(\Delta^{3/2}) \\ &= (2\,\Delta)^{1/2}2(-\tfrac{1}{2} + m) + 0(\Delta^{3/2}). \end{aligned} \tag{24}$$

Then, since

$$\sum_{m=0}^{\infty}\frac{(\frac{1}{2})_m}{m!}\rho_{23}^{2m} = (1 - \rho_{23}^2)^{-1/2}, \tag{25}$$

(22) gives the result,

$$\begin{aligned} w = 1 - \frac{2\sqrt{2}}{\pi}\Delta^{1/2} - \frac{2\sqrt{2}}{\pi}\delta^{1/2} \\ + \frac{8}{\pi^2}\Delta^{1/2}\,\delta^{1/2}(1 - e^{-2\tau})^{-1/2} \\ + \text{higher-order terms}. \end{aligned} \tag{26}$$

Finally (21) may be expanded and substituted into (20), along with (26). The first few terms cancel and (4) may be used; then

$$U_{\Delta\delta}(\tau) \sim \beta(\delta)(1 - e^{-2\tau})^{-1/2}. \tag{27}$$

Hence for a small but finite value of δ, $U_{\Delta\delta}(0)$ is infinite but $U_{\Delta\delta}(\tau)$ decreases monotonically as τ increases, ap-

[17] J. L. Lawson and G. E. Uhlenbeck, "Threshold Signals," McGraw-Hill Book Co., Inc., New York, N. Y., p. 57; 1950.

[18] J. A. McFadden, "Two expansions for the quadrivariate normal integral," *Biometrika*, vol. 47, pp. 325–333; December, 1960.

[19] Bateman Manuscript Project, "Higher Transcendental Functions," McGraw-Hill Book Co., Inc., New York, N. Y., vol. 1, p. 61, (14); 1953. This formula would ordinarily be valid only when $c > a + b$, but, since this series terminates, the restriction is unnecessary.

roaching the value $\beta(\delta)$. This limit is correct since $\beta(\delta)\delta$ is the probability of one or more crossings in an interval of length δ; the initial condition has vanishing influence as $\tau \to \infty$. The leading term (27) does not contain Δ.

Renewal Equations

If it is assumed that the lengths of successive axis-crossing intervals are statistically independent (*i.e.*, they form a renewal process), then certain well-known relations exist[20] between the Laplace transforms of $P_0(\tau)$, $U(\tau)$ and $r(\tau)$. If $u(s)$ and $f(s)$ are the Laplace transforms of $U(\tau)$ and $r(\tau)$, respectively, then it is easy to show that

$$u(s) = \frac{p_0(s)}{1 - p_0(s)}, \tag{28}$$

and

$$f(s) = \frac{1}{s} - \frac{2\beta}{s^2}\frac{1 - p_0(s)}{1 + p_0(s)}. \tag{29}$$

On the other hand, it was shown by Palmer[21] and McFadden[2,22] that for stationary Gaussian processes in which the autocorrelation function $\rho(\tau)$ possesses certain derivatives at the origin, the lengths of successive axis-crossing intervals cannot be independent. The Markov case, being somewhat unique, must be investigated separately.

The Laplace transform of $U_{\Delta\delta}(\tau)$ in (27) is, asymptotically,[13]

$$u_{\Delta\delta}(s) \sim \sqrt{\frac{2}{\pi}}\,\delta^{-1/2}\,\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2} + \frac{1}{2}\right)}; \tag{30}$$

whereas by (11),

$$\frac{p_\Delta(s)}{1 - p_\Delta(s)} = \sqrt{\frac{2}{\pi}}\,\Delta^{-1/2}\,\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2} + \frac{1}{2}\right)} + 0(1). \tag{31}$$

Thus, asymptotically for small δ, or Δ, a renewal equation similar to (28) is satisfied between $p_\delta(s)$ and $u_{\Delta\delta}(s)$, or between $p_\Delta(s)$ and $u_{\delta\Delta}(s)$.

$$u_{\Delta\delta}(s) \sim \frac{p_\delta(s)}{1 - p_\delta(s)}. \tag{32}$$

Consider next the autocorrelation function after clipping. By (1) and (21),

$$r(\tau) = \frac{2}{\pi}\sin^{-1}\left(e^{-|\tau|}\right). \tag{33}$$

It follows from integral tables[23] that the Laplace transform is

$$f(s) = \frac{1}{s} - \frac{1}{\pi s}B\left(\frac{s}{2} + \frac{1}{2}, \frac{1}{2}\right). \tag{34}$$

On the other hand, by (4) and (11),

$$\frac{2\beta(\Delta)}{s^2}\frac{1 - p_\Delta(s)}{1 + p_\Delta(s)} = \frac{1}{\pi s}B\left(\frac{s}{2} + \frac{1}{2}, \frac{1}{2}\right) + 0(\Delta^{1/2}). \tag{35}$$

Thus, asymptotically for small Δ, a renewal equation similar to (29) is satisfied, where β has been replaced by $\beta(\Delta)$ and $p_0(s)$ by $p_\Delta(s)$.

$$f(s) = \frac{1}{s} - \frac{2\beta(\Delta)}{s^2}\frac{1 - p_\Delta(s)}{1 + p_\Delta(s)} + 0(\Delta^{1/2}). \tag{36}$$

Application to the Filter-Clip-Filter Problem

The above theory of axis crossings has an application in the following problem. Let $\xi(t)$ be a stationary Gaussian Markov process, as defined previously, *i.e.*, the output of an RC low-pass filter (with RC = 1), when the input is stationary, white Gaussian noise. $x(t)$ is the output after $\xi(t)$ is infinitely clipped, as in (18). Now let $x(t)$ be the input to a second RC low-pass filter with RC = T, and let $y(t)$ be the final output. It is desired to find moments or, if possible, the distribution of $y(t)$. This may be called the "filter-clip-filter" problem.

Previously McFadden[24] has studied the distribution of the output of an RC filter when the input is a stationary binary random process. The lengths of the axis-crossing intervals of the input were assumed to be statistically independent and identically distributed.

If this method is applied to the filter-clip-filter problem, two difficulties arise: The first is the questionability of the assumption of the independence of the lengths of successive axis-crossing intervals. The second is the fact that $p_0(s) = 1$, causing some of the formulas to become indeterminate.

Nevertheless, proceeding formally with $p_\Delta(s)$ from (11), in place of $p_0(s)$, expressions for $E[y^2(t)]$ and $E[y^4(t)]$ have been obtained. When $\Delta \to 0$, these expressions agree with those obtained by other methods.

Complete results of the filter-clip-filter investigation will be published at a later date.

Conclusions

Although β, $P_0(\tau)$, and $U(\tau)$ do not exist (as previously defined) for a stationary Gaussian Markov process, asymptotic formulas for analogous quantities have been obtained. Even if strict independence of the lengths of successive axis-crossing intervals is not easily defined, the results suggest a type of asymptotic independence.

[20] McFadden, *op. cit.*, "The axis-crossing intervals of random functions II," (30) and (31), and the references cited therein.

[21] D. S. Palmer, "Properties of random functions," *Proc. Cambridge Phil. Soc.*, vol. 52, pp. 672–686; October, 1956.

[22] J. A. McFadden, "The fourth product moment of infinitely clipped noise," IRE Trans. on Information Theory, vol. IT-4, pp. 159–162; December, 1958.

[23] W. Gröbner and N. Hofreiter, "Integraltafel," Springer-Verlag, Vienna, Austria, vol. 2, p. 152, (5a); 1958.

[24] J. A. McFadden, "The probability density of the output of an RC filter when the input is a binary random process," IRE Trans. on Information Theory, vol. IT-5, pp. 174–178; December, 1959.

On Optimal Diversity Reception*

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Summary—The ideal probability-computing M-ary receiver is derived for a fading, noisy, multidiversity channel, in which the link fadings may be mutually correlated, as may the link noises. The results are interpreted in terms of block diagrams involving various filtering operations. Two special cases, those of very fast and very slow fading, are considered in detail.

I. Introduction

We shall examine in this paper the following hypothesis-testing problem, which is depicted in Fig. 1. One of a set of M waveforms, here denoted by their complex analytic representations,[1] $\xi_i(t)(i = 1, 2, \cdots, M)$, is transmitted into a channel which comprises L diversity links, *i.e.*, L ways by which the transmitted waveform can reach the receiver. These links are not deterministic, however; each is perturbed by two time-varying random disturbances, one multiplicative in nature and the other additive. Denoting the two disturbances in the lth of the L links by the complex analytic representations $\gamma_l(t)$ and $\nu_l(t)$, respectively, we accordingly write the output of the lth link as

$$\zeta_l(t) = \gamma_l(t)\xi_m(t) + \nu_l(t), \tag{1}$$

where we have assumed that $\xi_m(t)$ was transmitted. (Note that $|\gamma_l(t)|$ and $\tan^{-1}[Im\,\gamma_l(t)/Re\,\gamma_l(t)]$ represent, respectively, the random amplitude and phase modulations—*i.e.*, fading—suffered by the transmitted signal on traversing the lth link.[1]) The $\gamma_l(t)$'s may be correlated amongst themselves, as may be the $\nu_l(t)$'s; we shall assume, however, that the fadings and additive noises are statistically independent.

The receiver at the output of the channel has available L inputs of the form of (1), but does not know the value of the index m. It is called upon to guess the true value of m on the basis of its observations of these inputs. This guess is the receiver's output.

Clearly, the set of received waveforms, $\{\zeta_l(t)\}$, may be written as a vector (column matrix) $\mathbf{Z}(t)$, the lth component of which is $\zeta_l(t)$. If we similarly write the sets of functions $\{\gamma_l(t)\}$ and $\{\nu_l(t)\}$ as the stochastic vectors $\mathbf{\Gamma}(t)$ and $\mathbf{N}(t)$, we may write $\mathbf{Z}(t)$ as

$$\mathbf{Z}(t) = \xi_m(t)\mathbf{\Gamma}(t) + \mathbf{N}(t). \tag{2}$$

We may then define the task of the receiver as that of transforming the stochastic vector $\mathbf{Z}(t)$ into a scalar which may assume any one of M values.

Physical examples of communication channels of this type are numerous. Perhaps the most obvious is that of space diversity, where $\zeta_l(t)$ represents the signal received by way of the lth of L antennas. Again, with the mathematically unimportant insertion of known frequency shifts (which may be included for convenience in the $\gamma_l(t)$), the model may be made to correspond to a frequency-diversity situation in which $\zeta_l(t)$ represents the signal received over the lth of L nonoverlapping frequency bands. The model may also depict the time diversity of a resolvable multipath situation in which $\zeta_l(t)$ represents the signal received via the lth of L resolvable (*i.e.*, separable) paths of known modulation delay.[2]

In order to solve the problem just defined, we require certain well-established mathematical results, which are summarized in the following section.

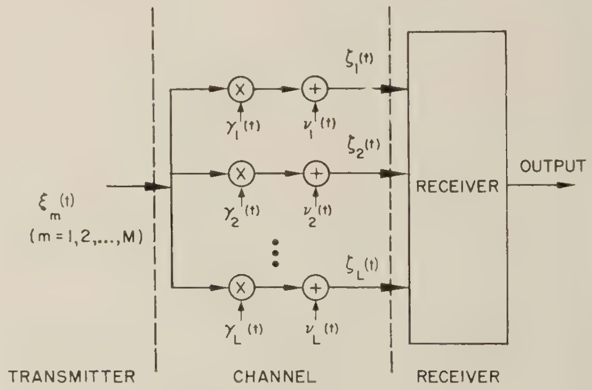

Fig. 1—The system under consideration.

II. Mathematical Preliminaries

A. Representation of Vector Stochastic Processes

Let $\mathbf{X}(t)$ be a finite-dimensional vector stochastic process with complex components, $x_i(t)$, all of zero mean, and let the covariance-function matrix[3] $\mathbf{K}(s, t) = E[\mathbf{X}(s)\mathbf{X}'^*(t)]$ be such that all of its components $E[x_i(s)x_j^*(t)]$ exist and are continuous on some finite square $[a \le s \le b, a \le t \le b]$.

* Received by the PGIT, August 24, 1960.
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[1] $Re\ \xi_i(t)$ is the actual ith physical waveform, and $Im\,\xi_i(t)$ is defined as the Hilbert transform of $Re\,\xi_i(t)$ [see (27) and (28) for an example of Hilbert transform relations]. For narrow-band waveforms, we may approximately identify $|\xi_i(t)|$ with the envelope, and $\tan^{-1}[Im\,\xi_i(t)/Re\,\xi_i(t)]$ with the phase, of the ith physical waveform. Cf., P. M. Woodward, "Probability and Information Theory, with Applications to Radar," McGraw-Hill Book Co., Inc., New York, N. Y.; 1953.

[2] G. L. Turin, "Communication through noisy, random-multipath channels," 1956 IRE Convention Record, pt. 4, pp. 154–166.

[3] A prime denotes "transpose"; an asterisk denotes "complex conjugate."

en the following statements hold;[4–9] most of these immediate extensions of their one-dimensional equivnts.[10]

1) The representation

$$\mathbf{X}(t) = \sum_k \alpha_k \boldsymbol{\Phi}_k(t) \tag{3}$$

nverges uniformly in the mean square to $\mathbf{X}(t)$ on the erval $a \leq t \leq b$, where the complex scalars α_k are en by

$$\alpha_k = \int_a^b \boldsymbol{\Phi}_k'^*(t)\mathbf{X}(t)\,dt, \tag{4}$$

d the $\boldsymbol{\Phi}_k(t)$ are orthonormalized vector eigenfunctions the matrix integral equation

$$\int_a^b \mathbf{K}(s,t)\boldsymbol{\Phi}(t)\,dt = \mu\boldsymbol{\Phi}(s), \qquad a \leq s \leq b. \tag{5}$$

2) By virtue of the orthonormality of the $\boldsymbol{\Phi}_k(t)$,

$$\int_a^b \boldsymbol{\Phi}_k'^*(t)\boldsymbol{\Phi}_l(t)\,dt = \delta_{kl}, \tag{6}$$

here δ_{kl} is the Kronecker delta. Further,

$$E(\alpha_k\alpha_l^*) = \begin{cases}\mu_k & k = l \\ 0\,, & k \neq l\end{cases}, \tag{7}$$

here μ_k is the eigenvalue of (5) corresponding to the ution $\boldsymbol{\Phi}_k(t)$.

3) A generalization of Mercer's theorem for one dinsion yields the representation

$$\mathbf{K}(s,t) = \sum_k \mu_k \boldsymbol{\Phi}_k(s)\boldsymbol{\Phi}_k'^*(t), \qquad a \leq s, t \leq b, \tag{8}$$

m which it follows that

$$\mu_k = \int_a^b \int_a^b \boldsymbol{\Phi}_k'^*(s)\mathbf{K}(s,t)\boldsymbol{\Phi}_k(t)\,ds\,dt. \tag{9}$$

4) Let $\mathbf{J}(s,t)$ be defined as the inverse of $\mathbf{K}(s,t)$ in the sense that if $\mathbf{F}(t)$ is a vector, and

$$\mathbf{G}(s) = \int_a^b \mathbf{K}(s,t)\mathbf{F}(t)\,dt, \tag{10}$$

then

$$\mathbf{F}(s) = \int_a^b \mathbf{J}(s,t)\mathbf{G}(t)\,dt. \tag{11}$$

(We may thus symbolically write

$$\int_a^b \mathbf{K}(s,u)\mathbf{J}(u,t)\,du = \mathbf{I}\,\delta(s-t), \tag{12}$$

where $\mathbf{I}$ is the unit matrix and $\delta(t)$ is the Dirac delta function.) Then, formally, $\mathbf{J}(s,t)$ has the representation

$$\mathbf{J}(s,t) = \sum_k \mu_k^{-1}\boldsymbol{\Phi}_k(s)\boldsymbol{\Phi}_k'^*(t), \qquad a \leq s, t \leq b. \tag{13}$$

A sufficient condition for $\mathbf{J}(s,t)$ to exist is that $\mathbf{K}(s,t)$ be positive definite [see (18), below].

5) If $\mathbf{K}_n(s,t)$ is the nth iteration of $\mathbf{K}(s,t)$, that is, if

$$\mathbf{K}_1(s,t) \equiv \mathbf{K}(s,t) \tag{14}$$

and

$$\mathbf{K}_n(s,t) = \int_a^b \mathbf{K}(s,u)\mathbf{K}_{n-1}(u,t)\,du, \qquad n \geq 2, \tag{15}$$

then we may write

$$\mathbf{K}_n(s,t) = \sum_k \mu_k^n \boldsymbol{\Phi}_k(s)\boldsymbol{\Phi}_k'^*(t), \qquad a \leq s, t \leq b. \tag{16}$$

Further, it is easily seen from (6) and (16) that

$$tr \int_a^b \mathbf{K}_n(s,s)\,ds = \sum_k \mu_k^n, \tag{17}$$

where "*tr*" denotes the matrix trace.

6) If $\mathbf{K}(s,t)$ is positive definite, *i.e.*, if

$$\int_a^b \int_a^b \mathbf{F}'^*(s)\mathbf{K}(s,t)\mathbf{F}(t)\,ds\,dt > 0 \tag{18}$$

holds for any $\mathbf{F}(t)$ which satisfies

$$\int_a^b \mathbf{F}'^*(t)\mathbf{F}(t)\,dt < \infty, \tag{19}$$

then $\mu_k > 0$, all k. Further, the set of vector eigenfunctions, $\{\boldsymbol{\Phi}_k(t)\}$, of (5) is complete, *i.e.*, any $\mathbf{F}(t)$ which satisfies (19) may be written in the form

$$\mathbf{F}(t) = \sum_k \beta_k \boldsymbol{\Phi}_k(t), \qquad a \leq t \leq b, \tag{20}$$

where

$$\beta_k = \int_a^b \boldsymbol{\Phi}_k'^*(t)\mathbf{F}(t)\,dt \tag{21}$$

are called the expansion coefficients of $\mathbf{F}(t)$.

[4] A. C. Zaanen, "Linear Analysis," North Holland Publishing ., Amsterdam, Netherlands; 1953.

[5] J. B. Thomas and L. A. Zadeh, "Note on an integral equation urring in the prediction, detection, and analysis of multiple e series," IRE Trans. on Information Theory, vol. IT–7, . 118–120; April, 1961.

[6] E. Wong, "Vector Stochastic Processes in Problems of Comnication Theory," Ph.D. Thesis, Princeton University, Prince, N. J.; May, 1959. See also, J. B. Thomas and E. Wong, "On statistical theory of optimum demodulation," IRE Trans. on formation Theory, vol. IT–6, pp. 420–425; September, 1960.

[7] J. K. Wolf, "On the Detection and Estimation Problem for ltiple Nonstationary Random Processes," Ph. D. Thesis, nceton University, Princeton, N. J.; October, 1959. See also B. Thomas and J. K. Wolf, "On the statistical detection problem multiple signals," IRE Trans. on Information Theory, to be blished.

[8] E. J. Kelly and W. L. Root, "Representations of Vectorlued Random Processes," Lincoln Lab., M. I. T., Lexington, ass., Group Rept. 55–21; March 7, 1960. Also, *J. Math. and Phys.*, . 39, pp. 211–216; October, 1960.

[9] A. V. Balakrishnan, "Estimation and detection theory for ltiple stochastic processes," *J. Math. Anal. and Appl.*, vol. 1, . 386–400; December, 1960.

[10] W. B. Davenport, Jr. and W. L. Root, "An Introduction to Theory of Random Signals and Noise," McGraw-Hill Book ., Inc., New York. N. Y., pp. 96–101 and Appendix 2; 1958.

Note, however, that whether or not the set $\{\mathbf{\Phi}_k(t)\}$ is complete, it follows from (8) that the representation

$$\int_a^b \mathbf{K}(s, t)\mathbf{F}(t)\, dt = \sum_k \mu_k \beta_k \mathbf{\Phi}_k(s), \qquad a \leq t \leq b \tag{22}$$

is valid, provided $\mathbf{F}(t)$ satisfies (19).

7) If $\mathbf{G}(t)$ is another vector for which expressions of the forms of (19), (20) and (21) hold, and we denote the expansion coefficients of $\mathbf{G}(t)$ by ϵ_k, then it follows from (6) that

$$\int_a^b \mathbf{F}'^*(t)\mathbf{G}(t)\, dt = \sum_k \beta_k^* \epsilon_k. \tag{23}$$

This is a generalized Parseval theorem.

8) If $\mathbf{K}(s, t)$ is positive semidefinite, *i.e.*, if (18) may also hold with an equality, then $\mu_k \geq 0$, all k, and we may define a "square root" of $\mathbf{K}(s, t)$:

$$\sqrt{\mathbf{K}(s, t)} \equiv \sum_k \sqrt{\mu_k}\, \mathbf{\Phi}_k(s)\mathbf{\Phi}_k'^*(t), \quad a \leq s, t \leq b. \tag{24}$$

We then have from (6) and (8)

$$\int_a^b \sqrt{\mathbf{K}(s, u)}\, \sqrt{\mathbf{K}(u, t)}\, du = \mathbf{K}(s, t). \tag{25}$$

All covariance-function matrices are at least positive semidefinite, and are very often positive definite.

B. A Class of Vector Stochastic Processes

Suppose that $\mathbf{X}(t)$ is a zero-mean vector stochastic process which may be written in the form

$$\mathbf{X}(s) = \int_a^b \mathbf{A}(s, t)\mathbf{Y}(t)\, dt, \tag{26}$$

where $\mathbf{A}(s, t)$ is a matrix of nonrandom functions and $\mathbf{Y}(t)$ is a wide-sense stationary vector stochastic process. Suppose further that $\mathbf{Y}(t)$ is analytic,[1] so we may write the Hilbert transform-pair

$$Im\, \mathbf{Y}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{Re\, \mathbf{Y}(\tau)}{t - \tau}\, d\tau \tag{27}$$

$$Re\, \mathbf{Y}(t) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{Im\, \mathbf{Y}(\tau)}{t - \tau}\, d\tau, \tag{28}$$

where the principal values of the integrals are implied. Then, using the assumed stationarity of $\mathbf{Y}(t)$, it is easily shown from (27) and (28) that[11]

$$E[Re\, \mathbf{Y}(s)\, Re\, \mathbf{Y}'(t)] = E[Im\, \mathbf{Y}(s)\, Im\, \mathbf{Y}'(t)] \tag{29}$$

and

$$E[Re\, \mathbf{Y}(s)\, Im\, \mathbf{Y}'(t)] = -E[Im\, \mathbf{Y}(s)\, Re\, \mathbf{Y}'(t)]. \tag{30}$$

[11] Note that (29)–(31) do not hold for *any* analytic vector $\mathbf{Y}(t)$; an example to which they do not apply is the one-dimensional vector $\exp[j(\omega t + \varphi)]$ where φ is *not* uniformly distributed.

That is,

$$E[\mathbf{Y}(s)\mathbf{Y}'(t)] = 0, \tag{31}$$

and hence

$$E[\mathbf{X}(s)\mathbf{X}'(t)] = \int_a^b \int_a^b \mathbf{A}(s, u)E[\mathbf{Y}(u)\mathbf{Y}'(v)]\mathbf{A}'(t, v)\, du\, dv = 0. \tag{32}$$

Then, if $\mathbf{X}(t)$ is expanded according to (3), we have from (4)

$$E(\alpha_k^2) = E\int_a^b \int_a^b \mathbf{\Phi}_k'^*(s)E[\mathbf{X}(s)\mathbf{X}'(t)]\mathbf{\Phi}_k^*(t)\, ds\, dt = 0, \tag{33}$$

so that

$$E[(Re\, \alpha_k)^2] = E[(Im\, \alpha_k)^2] = \tfrac{1}{2}\mu_k \tag{34}$$

and

$$E[Re\, \alpha_k\, Im\, \alpha_k] = 0. \tag{35}$$

Thus, not only are the expansion coefficients uncorrelated with one another [see (7)], but the real and imaginary parts of each are uncorrelated and have equal variances.

If we now assume that the real and imaginary parts of the component waveforms of the vector $\mathbf{X}(t)$ are jointly Gaussian, then it follows from (4) that the real and imaginary parts of the expansion coefficients are also jointly Gaussian. From (7), (34) and (35) we therefore have for the joint density distribution of $\{Re\, \alpha_k\}$ and $\{Im\, \alpha_k\}$:

$$pr\,[Re\, \alpha_1, Im\, \alpha_1, Re\, \alpha_2, Im\, \alpha_2, \cdots] = \prod_k \frac{1}{\pi\mu_k} \exp\left[-\frac{(Re\, \alpha_k)^2 + (Im\, \alpha_k)^2}{\mu_k}\right] = c\exp\left[-\sum_k \frac{|\alpha_k|^2}{\mu_k}\right], \tag{36}$$

where

$$c = \prod_k \frac{1}{\pi\mu_k}. \tag{37}$$

Letting $\mathbf{J}(s, t)$ be the inverse, in the sense of (13), of $\mathbf{K}(s, t)$ and applying relationships of the forms of (22) and (23) to the exponent of (36), we may rewrite the joint density of (36) as

$$pr\,[\mathbf{X}(t)] = c\exp\left[-\int_a^b \int_a^b \mathbf{X}'^*(s)\mathbf{J}(s, t)\mathbf{X}(t)\, ds\, dt\right], \tag{38}$$

provided $\mathbf{J}(s, t)$ exists. It should be carefully noted that the "density" $pr\,[\mathbf{X}(t)]$ of (38) is only a shorthand notation for the joint density of the real and imaginary parts of the expansion coefficients of $\mathbf{X}(t)$.

III. Formal Solution of the Problem

We may now establish a formal solution to the problem sed in the introductory section.[12] Adopting the view-int of Woodward and Davies[13] and Kotel'nikov,[14] we cognize that the task of the receiver is to calculate the *posteriori* probabilities, $Pr\ [\xi_m(t)/\mathbf{Z}(t)]$, that the mth nal was sent, given that the received vector $\mathbf{Z}(t)$ was served. The largest of these M probabilities is then und, and the corresponding value of m is given as the ceiver output in Fig. 1.

Now, the required *a posteriori* probabilities are, by yes' equality,

$$Pr\ [\xi_m(t)/\mathbf{Z}(t)] = \frac{P_m\, pr\ [\mathbf{Z}(t)/\xi_m(t)]}{pr\ [\mathbf{Z}(t)]}, \tag{39}$$

here P_m is the *a priori* probability of transmission of (t). The "densities" $pr\ [\mathbf{Z}(t)/\xi_m(t)]$ and $pr\ [\mathbf{Z}(t)]$ are ken in the sense used in the previous section. We sume that the *a priori* probabilities, P_m, are known the receiver; then, since $pr\ [\mathbf{Z}(t)]$ does not depend on m, e receiver's task reduces to the evaluation of the likeli-ods

$$\Lambda_m = pr\ [\mathbf{Z}(t)/\xi_m(t)], \tag{40}$$

., the probability "densities" of receiving $\mathbf{Z}(t)$, assum-g that $\xi_m(t)$ was transmitted.

As a first step toward finding an expression for these elihoods, let us rewrite (2) as

$$\mathbf{Z}(t) = \xi_m(t)\mathbf{\Gamma}_1(t) + \xi_m(t)\mathbf{\Gamma}_2(t) + \mathbf{N}(t). \tag{41}$$

ere we have split the transmission vector, $\mathbf{\Gamma}(t)$, into o parts, $\mathbf{\Gamma}_1(t)$ and $\mathbf{\Gamma}_2(t)$, the first random and the cond nonrandom. The latter is defined to be the mean $\mathbf{\Gamma}(t)$, *i.e.*, $\mathbf{\Gamma}_2(t) = E[\mathbf{\Gamma}(t)]$; it is assumed known to the ceiver.[15] Notice now that for the purpose of computing of (40), the receiver must assume that $\xi_m(t)$ was trans-tted. Under this assumption, it knows fully the second rm in (41). Let us therefore form a new vector,

$$\mathbf{W}(t) \equiv \mathbf{Z}(t) - \xi_m(t)\mathbf{\Gamma}_2(t), \tag{42}$$

ich is the random part of $\mathbf{Z}(t)$. The probability, as-ming $\xi_m(t)$ sent, that $\mathbf{Z}(t)$ is received, is then simply probability that $\xi_m(t)\mathbf{\Gamma}_1(t) + \mathbf{N}(t)$ can be equal to new vector defined in (42). That is,

$$\Lambda_m = pr\ [\xi_m(t)\mathbf{\Gamma}_1(t) + \mathbf{N}(t) = \mathbf{W}(t)/\xi_m(t)]. \tag{43}$$

In order to calculate the probability "density" of (43), we first consider the conditional "density", $pr\ [\mathbf{N}(t) = \mathbf{W}(t) - \xi_m(t)\mathbf{\Gamma}_1(t)/\xi_m(t),\ \mathbf{\Gamma}_1(t)]$, *i.e.*, the probability that the noise can take on the form $\mathbf{W}(t) - \xi_m(t)\mathbf{\Gamma}_1(t)$, where $\mathbf{\Gamma}_1(t)$ is temporarily assumed to be known. If we suppose that the noise vector is independent of $\mathbf{\Gamma}_1(t)$, and is a Gaussian process of the type described in section II-B,[16] we can, from (38), immediately write an expression for this conditional probability:

$$pr\ [\mathbf{N}(t) = \mathbf{W}(t) - \xi_m(t)\mathbf{\Gamma}_1(t)/\xi_m(t),\ \mathbf{\Gamma}_1(t)]$$
$$= c_N \exp\left\{-\int_0^T \int_0^T [\mathbf{W}(s) - \xi_m(s)\mathbf{\Gamma}_1(s)]'^* \cdot \mathbf{Q}(s,t)[\mathbf{W}(t) - \xi_m(t)\mathbf{\Gamma}_1(t)]\, ds\, dt\right\}. \tag{44}$$

Here $\mathbf{Q}(s, t)$ is the inverse, assumed to exist in the sense of (10)–(13), of the covariance-function matrix of $\mathbf{N}(t)$, c_N is a constant of the form of (37), and $(0, T)$ is the interval during which the receiver input is observed.

In order to simplify (44), let us make use of the fact that $\mathbf{Q}(s, t)$, being the inverse of a covariance-function matrix, has a square root in the sense of (25). Then, if we define two new vectors,

$$\mathbf{U}(s) = \int_0^T \sqrt{\mathbf{Q}(s,t)}\, \mathbf{W}(t)\, dt \tag{45}$$

and

$$\mathbf{V}(s) = \int_0^T \sqrt{\mathbf{Q}(s,t)}\, \xi_m(t)\mathbf{\Gamma}_1(t)\, dt, \tag{46}$$

we may rewrite the exponent of (44) as

$$-\int_0^T \mathbf{U}'^*(t)\mathbf{U}(t)\, dt - \int_0^T \mathbf{V}'^*(t)\mathbf{V}(t)\, dt + 2\, Re \int_0^T \mathbf{V}'^*(t)\mathbf{U}(t)\, dt. \tag{47}$$

In going from (44) to (46), we have made an obvious expansion of the integrand, and have invoked the relationship

$$\sqrt{\mathbf{Q}(t,s)} = (\sqrt{\mathbf{Q}(s,t)})'^*, \tag{48}$$

which follows easily from the properties of covariance-function matrices, hence their inverses and the square roots of these latter.

As a next step, let us expand $\mathbf{V}(t)$ in a series of the form of (3):

$$\mathbf{V}(t) = \sum_k \eta_k \mathbf{\Psi}_k(t). \tag{49}$$

[12] T. Kailath has also considered this problem, using a different hnique. See "Optimum Diversity Combiners," Research Lab. Electronics, M.I.T., Cambridge, Mass., Quart. Progr. Rept., pp. 3–200; July 15, 1960.

[13] P. M. Woodward and I. L. Davies, "Information theory and erse probability in telecommunications," *Proc. IEE*, vol. 99, III, pp. 37–44; March, 1952.

[14] V. A. Kotel'nikov, "The Theory of Optimum Noise Im-nity," McGraw-Hill Book Co., Inc., New York, N. Y.; 1959.

[15] Physically, $\mathbf{\Gamma}_1(t)$ may represent, say, a scatter-transmission de in the transmission medium, and $\mathbf{\Gamma}_2(t)$, a purely reflective or ractive mode of known properties.

[16] Such will be the case, for example, if the noises in the several links are correlated, wide-sense stationary processes. We then identify $\mathbf{Y}(t)$ in (26) with $\mathbf{N}(t)$ and let $\mathbf{A}(s, t)$ be a diagonal matrix of Dirac delta functions.

According to the results of section II-A, we then have that

$$\eta_k = \int_0^T \mathbf{\Psi}_k'^*(t)\mathbf{V}(t)\,dt, \tag{50}$$

and the $\mathbf{\Psi}_k(t)$ are orthonormalized vector eigenfunctions of

$$\int_0^T \mathbf{R}_m(s, t)\mathbf{\Psi}(t)\,dt = \sigma\mathbf{\Psi}(s), \qquad 0 \leq s \leq T, \tag{51}$$

where

$$\mathbf{R}_m(s, t) = E[\mathbf{V}(s)\mathbf{V}'^*(t)]. \tag{52}$$

Note that the dependence, through (46), of $\mathbf{R}_m(s, t)$ on $\xi_m(t)$ has been made explicit by use of a subscript.

If we define

$$\theta_k = \int_0^T \mathbf{\Psi}_k'^*(t)\mathbf{U}(t)\,dt, \tag{53}$$

and substitute (49) and (53) into (47) and the result into (44), we obtain

$$pr\,[\mathbf{N}(t) = \mathbf{W}(t) - \xi_m(t)\mathbf{\Gamma}_1(t)/\xi_m(t), \eta_1, \eta_2, \cdots]$$

$$= c_N \exp\left[-\int_0^T \mathbf{U}'^*(t)\mathbf{U}(t)\,dt - \sum_k |\eta_k|^2 + 2\,Re\sum_k \eta_k^*\theta_k\right]. \tag{54}$$

In (54), we have recognized that knowledge of the η_k's is equivalent to knowledge of $\mathbf{\Gamma}_1(t)$ if $\xi_m(t)$ is known.

Λ_m of (43) may now be obtained by averaging the conditional probability (54) over the η_k. If we assume that $\mathbf{V}(t)$ is a Gaussian vector process of the type described in section II-B,[17] then, following (36), we have for the joint distribution of the η_k:

$$pr\,[Re\,\eta_1, Im\,\eta_1, Re\,\eta_2, Im\,\eta_2, \cdots]$$

$$= c_F \exp\left[-\sum_k \frac{|\eta_k|^2}{\sigma_k}\right]. \tag{55}$$

The constant c_F is given by

$$c_F = \prod_k \frac{1}{\pi\sigma_k}, \tag{56}$$

and

$$\sigma_k = E[|\eta_k|^2] = \int_0^T\int_0^T \mathbf{\Psi}_k'^*(s)\mathbf{R}_m(s, t)\mathbf{\Psi}_k(t)\,ds\,dt \tag{57}$$

is the eigenvalue of (51) corresponding to the solution $\mathbf{\Psi}_k(t)$.

On multiplying (54) by (55) and integrating on $\{Re\,\eta_k\}$ and $\{Im\,\eta_k\}$, we finally obtain for the desired likelihoods:

$$\Lambda_m = c_N\left[\prod_k (1 + \sigma_k)\right]^{-1}$$

$$\cdot\exp\left[-\int_0^T \mathbf{U}'^*(t)\mathbf{U}(t)\,dt + \sum_k \frac{\sigma_k\,|\theta_k|^2}{1 + \sigma_k}\right]. \tag{58}$$

[17] This will occur, for example, if $\mathbf{\Gamma}_1(t)$ is composed of correlated, wide-sense stationary, Rayleigh-fading components; we may then identify $\sqrt{\mathbf{Q}(s, t)}\,\xi_m(t)$ with $\mathbf{A}(s, t)$ in (26) [cf. (46)].

This is a formal solution of the optimum-receiver problem, in which the parameter m appears on the right-hand side implicitly in $\mathbf{U}(t)$, σ_k and θ_k [see (42), (45), (52), (53) and (57)].

It is desirable, however, to eliminate the artifices of the mathematical derivation, *i.e.*, the eigenvectors $\mathbf{\Psi}_k(t)$ implicit in the θ_k's, and the eigenvalues σ_k. Such a procedure will replace mathematical artificialities with physically meaningful entities, and will obviate the necessity for solving the vector integral equation (51).

First, let us take the logarithm of both sides of (58)

$$ln\,\Lambda_m = l_n c_N - \sum_k ln\,(1 + \sigma_k)$$

$$- \int_0^T \mathbf{U}'^*(t)\mathbf{U}(t)\,dt + \sum_k \frac{\sigma_k\,|\theta_k|^2}{1 + \sigma_k}. \tag{59}$$

The first term on the right in this expression depends neither on the receiver input nor on the transmitted-waveform index, m; it therefore need not concern us further.

The second term does depend on m, although not on the received signal. If all the eigenvalues σ_k are less than unity,[18] we may write

$$B_m \equiv -\sum_k ln\,(1 + \sigma_k) = \sum_k\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\,\sigma_k^n < 0, \tag{60}$$

and, summing on k first with the use of a relationship of the form of (17), we obtain

$$B_m = \sum_k \frac{(-1)^n}{n}\,tr\int_0^T \mathbf{R}_{mn}(s, s)\,ds. \tag{61}$$

In (61), $\mathbf{R}_{mn}(s, t)$ is the nth iteration of $\mathbf{R}_m(s, t)$, defined in the manner of (14) and (15). A fuller discussion of the evaluation of B_m in terms of $\mathbf{R}_m(s, t)$ is given by Middleton for the single-diversity case; his discussion goes over completely to the multidiversity case, however, by replacing all covariance functions by covariance-function matrices, and taking the matrix trace of appropriate results.[19]

The remaining terms of (59), *i.e.*,

$$S_m = -\int_0^T \mathbf{U}'^*(t)\mathbf{U}(t)\,dt + \sum_k \frac{\sigma_k\,|\theta_k|^2}{1 + \sigma_k}, \tag{62}$$

depend both on the receiver input and on the index m. These terms may be rewritten as follows.

[18] In particular, this occurs at small channel signal-to-noise ratios, *i.e.*, when $\mathbf{N}(t)$ dominates $\xi_m(t)\,\mathbf{\Gamma}_1(t)$. For, suppose $\mathbf{N}(t)$ is multiplied by a factor of p. $\mathbf{V}(t)$ of (46) will then decrease by this same factor, as will the η_k of (50), and the σ_k of (57) will decrease as p^{-2}. Thus, if we let $p \to \infty$, we will have $\sigma_k \to 0$, all k. The ensuing series for B_m in (60) and (61) may therefore be expected to converge quite rapidly for very small signal-to-noise ratios.

[19] D. Middleton, "An Introduction to Statistical Communication Theory," McGraw-Hill Book Co., Inc., New York, N. Y.; 1960. See Section 17.1; in particular, the trace of the negative of Middleton's (17.19), evaluated for $\lambda = 1$, is the equivalent of (61) of the present paper. See also D. Middleton, "On the detection of stochastic signals in additive normal noise, I," IRE Trans. on Information Theory, vol. IT-3, pp. 86–121; June, 1957.

We first note that the matrix $\mathbf{I}\delta(s-t) + \mathbf{R}_m(s, t)$ is sitive definite in the sense of (18), even if $\mathbf{R}_m(s, t)$ is t. Hence, if $\mathbf{R}_m(s, t)$ in (51) were replaced by $(s - t) + \mathbf{R}_m(s, t)$, the new integral equation would rtainly possess a complete set of eigenfunctions. Further, ese eigenfunctions would include $\{\mathbf{\Psi}_k(t)\}$ as a subset, d the eigenvalues corresponding to this subset would $\{1 + \sigma_k\}$. By formally following (13),[20] we therefore n write an expansion for the inverse of $\mathbf{I}\delta(s-t) + \mathbf{R}_m(s,t)$ the form

$$(s, t) = \sum_k \frac{1}{1+\sigma_k} \mathbf{\Psi}_k(s)\mathbf{\Psi}_k'^*(t) + \mathbf{r}(s, t),$$

$$0 \leq s, t \leq T, \qquad (63)$$

here the second term, $\mathbf{r}(s, t)$, is similar in form to the st, but includes only those eigenfunctions of the new tegral equation which are not in, hence are orthogonal , $\{\mathbf{\Psi}_k(t)\}$. $\mathbf{r}(s, t)$ is clearly zero if $\mathbf{R}_m(s, t)$ is positive finite.

Now, following (22), we may write

$$\int^T \mathbf{R}_m(s, t)\mathbf{U}(t)\,dt = \sum_k \sigma_k\theta_k\mathbf{\Psi}_k(s), \quad 0 \leq s \leq T, \qquad (64)$$

here the θ_k are as in (53). Then, defining a new vector,

$$\hat{\mathbf{V}}(s) = \int_0^T \int_0^T \mathbf{P}(s, u)\mathbf{R}_m(u, t)\mathbf{U}(t)\,du\,dt, \qquad (65)$$

d using (63) and (64) in this definition, we have by rect calculation (using the orthonormality of the $\mathbf{\Psi}_k(t)$):

$$\hat{\mathbf{V}}(s) = \sum_k \frac{\sigma_k\theta_k}{1+\sigma_k}\mathbf{\Psi}_k(s), \qquad 0 \leq s \leq T. \qquad (66)$$

se of (53) and (66) easily shows that

$$\int_0^T \hat{\mathbf{V}}'^*(t)\mathbf{U}(t)\,dt = \sum_k \frac{\sigma_k \mid \theta_k \mid^2}{1+\sigma_k}, \qquad (67)$$

(62) finally becomes

$$S_m = -\int_0^T \mathbf{U}'^*(t)\mathbf{U}(t)\,dt + \int_0^T \hat{\mathbf{V}}'^*(t)\mathbf{U}(t)\,dt. \qquad (68)$$

his last expression, as we shall see, has a most interesting terpretation.

Eq. (68) may be rewritten in another form of interest invoking the inverse relationship between $\mathbf{P}(s, t)$ and $(s - t) + \mathbf{R}_m(s, t)$ to write [see (12)]:

$$(s) = \int_0^T \int_0^T \mathbf{P}(s, u)[\mathbf{I}\ \delta(u - t) + \mathbf{R}_m(u, t)]\mathbf{U}(t)\,du\,dt. \qquad (69)$$

r, insertion of (65) and (69) into (68) then yields

$$S_m = -\int_0^T \int_0^T \mathbf{U}'^*(s)\mathbf{P}(s, t)\mathbf{U}(t)\,ds\,dt. \qquad (70)$$

[20] We say "formally" since the new kernel does not satisfy the nditions imposed at the beginning of section II–A. The pertinent ults do carry over to the new kernel, however.

A third useful expression for S_m arises from letting

$$\mathbf{T}(s) = \int_0^T \sqrt{\mathbf{P}(s, t)}\,\mathbf{U}(t)\,dt, \qquad (71)$$

where $\sqrt{\mathbf{P}(s, t)}$ is defined in the manner of (25). We recognize that $\sqrt{\mathbf{P}(s, t)} = [\sqrt{\mathbf{P}(t, s)}]'^*$, whence we may easily show that (70) may be expressed as

$$S_m = -\int_0^T \mathbf{T}'^*(t)\mathbf{T}(t)\,dt. \qquad (72)$$

To recapitulate the results of this section, we recall that the task of the receiver is to find the value of m for which (39) is the largest. This may be done by finding the value of m which maximizes the quantity $\ln P_m + \ln \Lambda_m$, or, equivalently [*cf.* (59), (60) and (62)], which maximizes $S_m + B_m'$, where

$$B_m' = \ln P_m - \sum_k \ln(1 + \sigma_k) = \ln P_m + B_m. \qquad (73)$$

The biases, B_m', of (73) do not depend on the received signal; they may be calculated once and for all by means of (60) or (61). S_m, which does depend on the received signal, may be calculated by means of (62), (68), (70) or (72). Let us now consider physical interpretations of these mathematical results.

IV. Interpretations of the Results

In the foregoing analysis, we have often encountered two types of operation:

$$\mathbf{Y}(s) = \int_0^T \mathbf{H}(s, t)\mathbf{X}(t)\,dt \qquad (74)$$

and

$$\int_0^T \mathbf{X}'^*(t)\mathbf{Y}(t)\,dt. \qquad (75)$$

Let us therefore consider these in some detail.

The first is a linear operation on $\mathbf{X}(t)$, which may be interpreted in terms of a "matrix" filter. This filter has, say, q inputs, $x_j(t)(j = 1, \cdots, q)$, which are the components of $\mathbf{X}(t)$, and q outputs, $y_i(s)(i = 1, \cdots, q)$, which are the components of $\mathbf{Y}(s)$. From (74), these inputs and outputs are related by

$$y_i(s) = \sum_{j=1}^q \int_0^T h_{ij}(s, t)x_j(t)\,dt, \qquad (76)$$

where $h_{ij}(s, t)$ is the ijth element of the $q \times q$ matrix $\mathbf{H}(s, t)$, and represents a time-varying impulse response function giving the influence of the jth input on the ith output.

Note that (74) and (76), interpreted literally, do not always represent a physical operation, for we have generally been considering cases in which $\mathbf{X}(t)$, $\mathbf{Y}(t)$ and $\mathbf{H}(s, t)$ are complex. However, in the physical interpretations of our results which follow, we shall find that we are actually

only interested in the real parts of expressions of the form of (74), *i.e.*, in

$$Re\,\mathbf{Y}(s) = \int_0^T Re\,\mathbf{H}(s,t)\,Re\,\mathbf{X}(t)\,dt - \int_0^T Im\,\mathbf{H}(s,t)\,Im\,\mathbf{X}(t)\,dt. \tag{77}$$

Further, we shall see that in all cases we have considered, $\mathbf{H}(s, t)$ is "analytic" in the sense that $\mathbf{H}'^*(t, s) = \mathbf{H}(s, t)$ and

$$Im\,\mathbf{H}(s,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{Re\,\mathbf{H}(\sigma,t)}{s-\sigma}\,d\sigma = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{Re\,\mathbf{H}(s,\tau)}{t-\tau}\,d\tau, \tag{78}$$

where the principal values of the integrals are implied.

Then, if $\mathbf{X}(t)$ is analytic in the sense of (27) and (28), it is easy to show by direct calculation that the two terms in (77) are approximately equal, *i.e.*,[21]

$$Re\,\mathbf{Y}(s) \cong \int_0^T [2\,Re\,\mathbf{H}(s,t)]\,Re\,\mathbf{X}(t)\,dt. \tag{79}$$

That is, the real part of the output may be calculated through the use only of the real part of the input and the real part of the filter matrix. We shall henceforth depict an operation of the form of (79) as in Fig. 2, with the understanding that only the real parts of all complex quantities are meant.

In the case of the second type of operation, (75), we shall again find that in all cases it is the real part,

$$\int_0^T Re\,\mathbf{X}'(t)\,Re\,\mathbf{Y}(t)\,dt + \int_0^T Im\,\mathbf{X}'(t)\,Im\,\mathbf{Y}(t)\,dt, \tag{80}$$

in which we are interested. If $\mathbf{X}(t)$ and $\mathbf{Y}(t)$ are analytic, the two terms in (80) are approximately equal,[21] so we may write

$$Re\int_0^T \mathbf{X}'^*(t)\mathbf{Y}(t)\,dt \cong 2\int_0^T Re\,\mathbf{X}'(t)\,Re\,\mathbf{Y}(t)\,dt = 2\int_0^T\left[\sum_{i=1}^{q} Re\,x_i(t)\,Re\,y_i(t)\right]dt. \tag{81}$$

We shall depict (81) schematically as in Fig. 3, with the understanding that it is the real parts of the vectors $\mathbf{X}(t)$ and $\mathbf{Y}(t)$ which we must actually multiply together component by component.

With these points in mind, we may proceed to draw block diagrams of the ideal receiver. We seek to compute the quantity $S_m + B_m'$ for each value of m, where S_m may be obtained from the receiver input vector $\mathbf{Z}(t)$ by the sequence of operations (42), (45), (65) and (68), or by the sequence (42), (45), (71), (72). Note that each of these sequences terminates in the evaluation of an integral of the form of (75). Further, since it may be shown[22]

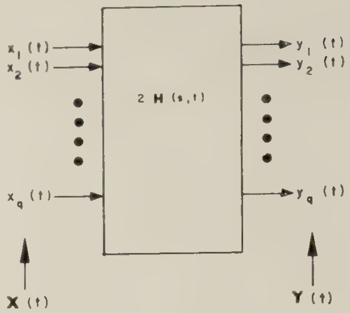

Fig. 2—A matrix filter.

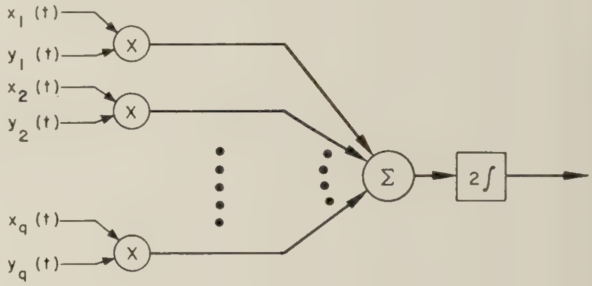

Fig. 3—A vector correlator.

that $\mathbf{U}(t)$, $\hat{\mathbf{V}}(t)$ and $\mathbf{T}(t)$ in these integrals are analytic and since S_m must of necessity be real, the evaluation may be performed using real parts, in the manner of (81). But the vectors $\mathbf{U}(t)$, $\hat{\mathbf{V}}(t)$ and $\mathbf{T}(t)$ are ultimately derived from $\mathbf{Z}(t)$ by sequences of complex matrix filtering of the form of (74); it is shown in Appendix I that the desired real parts of these vectors may be computed by corresponding sequences of real filtering operations of the form of (79). Therefore, following the convention adopted in Figs. 2 and 3, we may depict the computation of $S_m + B_m'$ as in Fig. 4 if the sequence of operations (42), (45), (65), (68) is used, and as in Fig. 5 if the sequence (42), (45), (71), (72) is used. In Fig. 4 we have used the notation [see (65)]

$$\mathbf{O}(s,t) = -4\int_0^T \mathbf{P}(s,u)\mathbf{R}_m(u,t)\,du, \tag{82}$$

and in Fig. 5 we have combined the filtering operations of (45) and (71) by letting

$$\mathbf{D}(s,t) = 4\int_0^T \sqrt{\mathbf{P}(s,u)}\,\sqrt{\mathbf{Q}(u,t)}\,du. \tag{83}$$

Since it is understood in these block diagrams that the real parts of all complex quantities are meant, we note in particular that the receiver input in both cases is $Re\,\mathbf{Z}(t)$, the actual set of received physical waveforms.

Fig. 4 has a particularly interesting and enlightening interpretation in terms of well-known results for optimum correlation reception through a channel disturbed only by additive, white, Gaussian noise.[13,14] In our case, of course, the additive disturbance in $\mathbf{Z}(t)$ (*i.e.*, $\mathbf{N}(t)$), although Gaussian, is not white; but it is easy to see that the filter $\sqrt{\mathbf{Q}(s,t)}$ has the effect of whitening this additive

[21] This statement, as well as others involved in the physical interpretation of the optimum-receiver equations, is an approximation, due to the fact that a finite time interval is being considered, rather than an infinite one. However, the statements are good approximations in the case most of interest, when the signals are narrow-band, *i.e.*, have center frequencies large compared to $1/T$. *Cf.* the discussion of truncation in Appendix III.

[22] See Appendix I.

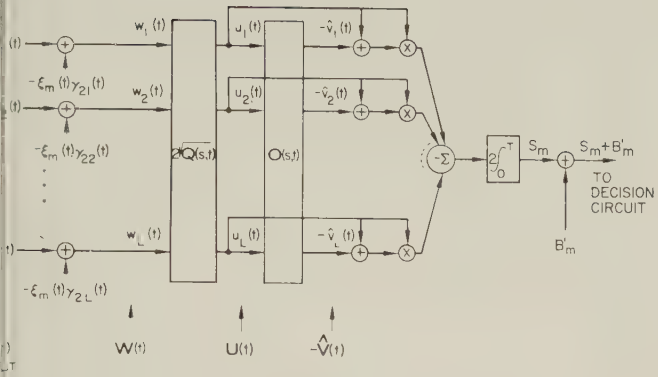

Fig. 4—The computation of $S_m + B_m'$.

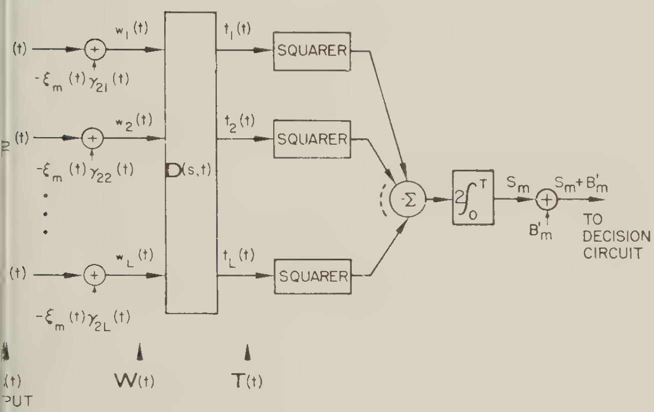

Fig. 5—Another way of computing $S_m + B_m'$.

sturbance. For, note from (41), (42), (45) and (46) at we may write the output of this filter as

$$\mathbf{U}(s) = \mathbf{V}(s) + \int_0^T \sqrt{\mathbf{Q}(s, t)}\, \mathbf{N}(t)\, dt, \tag{84}$$

here the first term on the right is due to the transmitted gnal, and the second term is an additive noise. The variance-function matrix of this additive noise is[23]

$$\int_0^T \int_0^T \sqrt{\mathbf{Q}(s, u)}\, E[\mathbf{N}(u)\mathbf{N}'^*(v)] \cdot (\sqrt{\mathbf{Q}(s, v)})'^*\, du\, dv = \mathbf{I}\delta(s - t). \tag{85}$$

hat is, the elements of the noise component of the vector (t) are *stationary*, *uncorrelated* (and hence, because ey are Gaussian, *independent*), and have identical *white* wer spectra.

Were this noise component the only disturbance in (t), we should expect that the remainder of the receiver ould be a multidiversity analog of a correlation receiver,[13,14] in which $\mathbf{U}(t)$ would be correlated with its gnal component, $\mathbf{V}(t)$, which would be known to the ceiver. Unfortunately, the signal component is *not* nown to the receiver. Similar work of Price[24] and Kailath,[12,25] suggests, however, that the receiver makes up for this lack by *estimating* $\mathbf{V}(t)$. This indeed turns out to be the case; it is shown in Appendix II that the output of the filter $\mathbf{O}(s, t)$ in Fig. 4, *i.e.*, $-\hat{\mathbf{V}}(t)$, is an optimum estimate of $-\mathbf{V}(t)$ in both the maximum-probability and minimum-variance senses.

Thus, as Price[24] and Kailath[12,25] have found in other cases, the optimum receiver of Fig. 4 is, after all, an extension of that of Woodward and Davies:[13] after a noise-whitening operation on $\mathbf{W}(t)$ to obtain $\mathbf{U}(t)$, the receiver performs a correlation operation, given by the second term on the right in (68), in which, for lack of having the true signal component of $\mathbf{U}(t)$ available to correlate with $\mathbf{U}(t)$, an estimate of this signal component is used. The first term on the right in (68) is analogous to the received-signal energy term in the Woodward-Davies receiver.

A final interpretive point is of great importance. It is clear that the signal component, $\xi_m(t)\mathbf{\Gamma}(t)$, of $\mathbf{Z}(t)$ may alternatively be interpreted not as the result of transmitting a known signal through a random medium, but as a stochastic signal with known statistics. Thus, for example, in the binary case we could take $\xi_1(t) \equiv 0$ and $\xi_2(t) \equiv 1$; then the problem we have been considering reduces to the detection of a random signal vector, $\mathbf{\Gamma}(t)$, in the presence of random noise.[19,24,26] From this point of view, (72) is seen to be related to a result of Wolf.[7]

V. Examples

In order to obtain further insight into the nature of the solutions depicted in Figs. 4 and 5, we consider below two special cases. The first exemplifies a receiver of the form of Fig. 5, and the second, one of the form of Fig. 4.

A. Very Fast Fading

Let us suppose for simplicity that the input noise vector $\mathbf{N}(t)$ is composed of independent, stationary, white noises; this is no great restriction on generality, since we have already seen that the receiver's noise-whitening filter would establish this state if it were not so. We may, for our purposes write the noise covariance-function matrix for this case approximately as (see Appendix III)

$$E[\mathbf{N}(s)\mathbf{N}'^*(t)] = 2\begin{bmatrix} N_{01} & & & \\ & N_{02} & & 0 \\ & & \ddots & \\ 0 & & & N_{0L} \end{bmatrix} \delta(s - t), \tag{86}$$

where N_{0l} is the single-ended noise power density in the lth diversity link. Then

$$\mathbf{Q}(s, t) = \mathbf{q}\ \delta(s - t), \tag{87}$$

[23] Eq. (85) follows immediately from (48) and from the fact that $\mathbf{Q}(s, t)$ is the inverse, in the sense of (12), of $\sqrt{E[\mathbf{N}(s)\, \mathbf{N}'^*(t)]}$.

[24] R. Price, "Optimum detection of random signals in noise with plications to scatter-multipath communication, I," IRE Trans. Information Theory, vol. IT-2, pp. 125–135; December, 1956.

[25] T. Kailath, "Correlation detection of signals perturbed by a random channel," IRE Trans. on Information Theory, vol. IT-6, pp. 361–366; June, 1960.

[26] R. C. Davis, "The detectability of random signals in the presence of noise," IRE Trans. on Information Theory, vol. IT-3, pp. 52–62; March, 1954.

where

$$\mathbf{q} = \tfrac{1}{2}\begin{vmatrix} N_{01}^{-1} & & & 0 \\ & N_{02}^{-1} & & \\ & & \ddots & \\ 0 & & & N_{0L}^{-1} \end{vmatrix}. \tag{88}$$

From (45) and (46) we therefore have

$$\mathbf{V}(s) = \sqrt{\mathbf{q}}\, \xi_m(s)\mathbf{\Gamma}_1(s) \tag{89}$$

and

$$\mathbf{U}(s) = \sqrt{\mathbf{q}}\, \mathbf{W}(s). \tag{90}$$

If we now make the further simplifying assumption that $\mathbf{\Gamma}_1(t)$ is stationary, then $\mathbf{R}_m(s, t)$ of (52) is of the form

$$\mathbf{R}_m(s, t) = \xi_m(s)\xi_m^*(t)\, \sqrt{\mathbf{q}}\, \mathbf{G}(s - t)\, \sqrt{\mathbf{q}}, \tag{91}$$

where

$$\mathbf{G}(s - t) = E[\mathbf{\Gamma}_1(s)\mathbf{\Gamma}_1'^*(t)]. \tag{92}$$

In order to define the condition for fast fading, we assume that $\mathbf{\Gamma}_1(t)$ varies so very much faster than the transmitted signal, $\xi_m(t)$, that whenever $\mathbf{\Gamma}_1(t)$ and $\xi_m(t)$ appear in a product, as in (91), we may make the approximation (as far as the $\xi_m(t)$ are concerned)[27]

$$\mathbf{G}(s - t) \cong \mathbf{g}\ \delta(s - t), \tag{93}$$

where $\mathbf{g}$ is a Hermitian matrix of constants, the klth element of which is twice the cross-power density of the fadings in the kth and lth links. Using (93), (91) becomes

$$\mathbf{R}_m(s, t) \cong \xi_m(s)\xi_m(t)\mathbf{C}\ \delta(s - t), \tag{94}$$

where $\mathbf{C} = \sqrt{\mathbf{q}}\ \mathbf{g}\ \sqrt{\mathbf{q}}$. Further,

$$\mathbf{I}\ \delta(s - t) + \mathbf{R}_m(s, t) \cong [\mathbf{I} + \xi_m(s)\xi_m^*(t)\mathbf{C}]\ \delta(s - t), \tag{95}$$

the inverse of which is [see (10)–(12)]

$$\mathbf{P}(s, t) = [\mathbf{I} + \xi_m(s)\xi_m^*(t)\mathbf{C}]^{-1}\ \delta(s - t). \tag{96}$$

We now use (70) to calculate S_m:

$$S_m = -\int_0^T \sqrt{\mathbf{q}}\ \mathbf{W}'^*(t)[\mathbf{I} + |\ \xi_m(t)\ |^2\mathbf{C}]^{-1}\mathbf{W}(t)\ \sqrt{\mathbf{q}}\ dt. \tag{97}$$

To investigate further the nature of (97), let us now assume that $\mathbf{g}$ is diagonal, *i.e.*, that the link fadings are independent. Then (97) becomes

$$S_m = -\sum_{l=1}^{L} \frac{1}{2N_{0l}} \int_0^T \frac{|\ w_l(t)\ |^2}{1 + (g_l\ |\ \xi_m(t)\ |^2/N_{0l})}\, dt, \tag{98}$$

where $w_l(t)$ is the lth element of $\mathbf{W}(t)$ and $2g_l$ is the lth diagonal element of $\mathbf{g}$. Recall from (42) that $w_l(t)$ is the random part of the lth receiver input, the known signal component having been removed. In the fast-fading case we have been considering, it is clear that the phase of this $w_l(t)$ bears no relationship to the phase of the transmitted signal, since signal phase has been randomized instant by instant by the transmission medium. Thus, we expect that the phase of $w_l(t)$, carrying no information about the transmitted signal, will not appear in the optimum-receiver expression when the fadings are independent.[28] This expectation is verified by (98), which is purely energetic in nature: only the instantaneous powers, $|\ w_l(t)\ |^2$, of the received signals enter, suitably weighted. Note that the weighting functions, $-[1 + (g_l\ |\ \xi_m(t)\ |^2/N_{0l})]^{-1}$, increase monotonically (although weakly) with the instantaneous signal-to-noise ratios,[29] $g_l\ |\ \xi_m(t)\ |^2/N_{0l}$, so that the stronger diversity links are emphasized, and these at the most favorable instants.

There remains the calculation of the biases, B_m', of (73). Note that these depend only on the *a priori* transmission probabilities, P_m, and on the eigenvalues, σ_k, of the integral equation (51). In the present case, the kernel of the equation is given by (91), where we have assumed that as far as the $\xi_m(t)$ are concerned, $\mathbf{G}(s - t)$ is approximately as in (93). Therefore, we may write the integral equation approximately as

$$|\ \xi_m(s)\ |^2 \int_0^T \sqrt{\mathbf{q}}\ \mathbf{G}(s - t)\ \sqrt{\mathbf{q}}\ \mathbf{\Psi}(t)\ dt = \sigma\mathbf{\Psi}(s), \qquad 0 \le s \le T. \tag{99}$$

Notice in particular that the solutions to this equation, hence the σ_k, depend only on the modulus of $\xi_m(t)$, not on its argument. Thus, the biases in the present case depend only on the amplitude modulations of the transmitted signals, not on their phases. If all the signals are *a priori* equiprobable and have identical envelopes, then all the biases are the same, and the receiver may make its decision solely on the basis of comparison of the S_m of (97) or (98).

We have already seen[18] that at small channel signal-to-noise ratios the series (61) for the term B_m in the bias B_m' may be expected to converge rapidly. In the event that the first term of the series suffices, we have for the present case [see (91)]

$$B_m \cong -\sqrt{\mathbf{q}}\ \mathbf{G}(0)\ \sqrt{\mathbf{q}} \int_0^T |\ \xi_m(s)\ |^2\ ds$$

$$= -2E_m\ \sqrt{\mathbf{q}}\ \mathbf{G}(0)\ \sqrt{\mathbf{q}}, \tag{100}$$

where E_m is the energy in the mth transmitted signal.

[27] Note that (93) assumes that, if there is link-to-link correlation of fading, the correlation only exists at simultaneous instants of time.

Strictly speaking, the covariance-function matrix of (93) violates the hypotheses of the mathematical formalism upon which our solutions are based. One may nonetheless justify its use by an argument of physical continuity: if a reasonable result is obtained by formally inserting (93) into the optimum-receiver equations, this must be an approximation to the result which would be obtained by using any *valid* covariance-function matrix to which (93) is an approximation.

[28] Of course, when the fadings are dependent, the *relative* phases of the $w_l(t)$'s will appear.

[29] g_l is the power density of the fading in the lth link, so $g_l|\ \xi_m(t)|^2$ is a measure of the instantaneous power of the signal component at the lth receiver input. N_{0l} is the noise power density in the lth link.

hus, at very small signal-to-noise ratios the biases are qual and may be eliminated from the receiver if the ransmitted signals are *a priori* equally probable and ave equal energies.

As a very simple example of the application of the bove results, let us consider a problem in radiometry: he detection of a wide-band random signal in the presence f thermal noise. For this case, as we have noted before, e may reverse the roles of $\mathbf{\Gamma}(t)$ and $\xi_m(t)$ in (2), associatng $\mathbf{\Gamma}(t)$ with the random signal. We then let $\xi_1(t) \equiv 0$ orrespond to the null hypothesis (noise only), and we ay let $\xi_2(t)$ correspond to some arbitrary waveform ith which we propose to modulate $\mathbf{\Gamma}(t)$ *prior* to its erturbation by $\mathbf{N}(t)$, the receiver's thermal noise.[30] f we consider for simplicity only the single-diversity ase, and let $\mathbf{\Gamma}_2(t) \equiv 0$ (no nonrandom component in $\mathbf{\Gamma}(t)$), we have from (42) and (98):

$$S_2 - S_1 = \frac{1}{2N_0}\int_0^T \frac{|\zeta(t)|^2}{1 + (N_0/g\,|\xi_2(t)|^2)}\,dt. \qquad (101)$$

hat is, the optimum radiometer must compute the orrelation between the squared envelope of the effective eceived waveform, $\zeta(t)$, and a monotone-increasing funcon of the envelope of the modulating waveform, $\xi_2(t)$ t small signal-to-noise ratios, this function is just $|\xi_2(t)|^2/N_0$). The quantity computed is compared with threshold determined by the biases, and it is decided at a signal is present if the threshold is exceeded.

. Very Slow Fading

We now consider the opposite extreme, in which $\mathbf{\Gamma}(t)$ aries so very much more slowly than $\xi_m(t)$ that we may nore its time dependence and denote it simply by $\mathbf{\Gamma}_1$. Ve shall again assume for simplicity that $\mathbf{N}(t)$ is comosed of independent, stationary, white noises, so (87)–(92) ill obtain, with $\mathbf{\Gamma}_1(t)$ and $\mathbf{G}(s-t)$ in (89), (91) and (92) eplaced by $\mathbf{\Gamma}_1$ and $\mathbf{G}$. In particular, if we let

$$\mathbf{M} = \sqrt{\mathbf{q}}\,\mathbf{G}\,\sqrt{\mathbf{q}}, \qquad (102)$$

hen

$$\mathbf{R}_m(s, t) = \xi_m(s)\xi_m^*(t)\mathbf{M}. \qquad (103)$$

It may easily be verified from (12) that the inverse of $(s-t) + \mathbf{R}_m(s, t)$ is

$$(s, t) = \mathbf{I}\,\delta(s-t) - \xi_m(s)\xi_m^*(t)(\mathbf{M}^{-1} + 2E_m\mathbf{I})^{-1}, \qquad (104)$$

here, as in (100), E_m is the energy in $\xi_m(t)$. On placing 04) into (70), we obtain

$$= -\int_0^T \mathbf{U}'^*(t)\mathbf{U}(t)\,dt + \mathbf{Y}'^*(\mathbf{M}^{-1} + 2E_m\mathbf{I})^{-1}\mathbf{Y}, \qquad (105)$$

here we have set

$$\mathbf{Y} = \int_0^T \xi_m^*(t)\mathbf{U}(t)\,dt. \qquad (106)$$

[30] R. H. Dicke, "The measurement of thermal radiation at microve frequencies," *Rev. Sci. Instr.*, vol. 17, pp. 268–275; July, 1946.

Note that (105) is of the form of (68), and is hence illustrated by Fig. 4.

As for the biases, it is shown in Appendix IV that the term B_m in (73) is

$$B_m = -ln\,|\,\mathbf{I} + 2E_m\mathbf{M}\,|, \qquad (107)$$

from which it is clear that the biases depend only on the energies of the signals and on their *a priori* probabilities.

Let us now further specialize the results in (105) and (107) by assuming that the link fadings are uncorrelated. Then $\mathbf{M}$ of (102) is diagonal, its lth diagonal element being, say, $\rho_l/2N_0$, where $\rho_l = E[\,|\gamma_{1l}|^2\,]$ is the mean-square envelope transmission strength of the random part of the lth transmission link. Using (90) and (106) in (105), we then have, for independent links,

$$S_m = \sum_{l=1}^{L}\left[-\frac{1}{2N_{0l}}\int_0^T |w_l(t)|^2\,dt + \frac{(\rho_l/4N_{0l}^2)\left|\int_0^T \xi_m^*(t)w_l(t)\,dt\right|^2}{1 + (\rho_l E_m/N_{0l})}\right]. \qquad (108)$$

Further, from (107),

$$B_m = -\sum_{l=1}^{L} ln\left(1 + \frac{\rho_l E_m}{N_{0l}}\right). \qquad (109)$$

Recall from (42) that $w_l(t)$ is related to the receiver inputs $\zeta_l(t)$ by

$$w_l(t) = \zeta_l(t) - \gamma_{2l}\xi_m(t), \qquad (110)$$

where γ_{2l} is the lth component of $\mathbf{\Gamma}_2$, the nonrandom part of the transmission vector.[15] On using (110), and assuming equal noises, $N_{0l} = N_0$, all l, (108) reduces to a result given in a previous paper.[31] In that paper it was shown that the physical operations corresponding to (108) comprise matched filtering, sampling, and a combination of coherent and noncoherent detection.

In particular, if $\rho_l = 0$, all l (*i.e.*, there is no random transmission component), (108) reduces to

$$S_m = -\sum_{l=1}^{L}\frac{1}{2N_{0l}}\int_0^T |\zeta_l(t)|^2\,dt - \sum_{l=1}^{L}\frac{|\gamma_{2l}|^2 E_m}{N_{0l}} + Re\int_0^T \xi_m^*(t)\left[\sum_{l=1}^{L}\frac{\gamma_{2l}^*\zeta_l(t)}{N_{0l}}\right]dt. \qquad (111)$$

In (111), the first term is independent of m and may be neglected; the second does not depend on the received signal, and contributes only to the bias. The third term corresponds to the optimal linear diversity combiner of Brennan:[32] the received waveforms $\zeta_l(t)$ are multiplied by the complex numbers γ_{2l}^*/N_{0l}, the phases of which place the signal components of the $\zeta_l(t)$—*viz.*, $\gamma_{2l}\xi_m(t)$ [see (1)]—in phase coherence, and the magnitudes of

[31] Turin, *op. cit.*, eq. (24). Identify $2\sigma_i^2$, $\alpha_i \exp(j\delta_i)$ and $\psi_m(\tau_i)$ in the cited paper with ρ_l, γ_{2l} and $\int_0^T \xi_m^*(t)\,\zeta_l(t)\,dt$, respectively, of the present paper.

[32] D. G. Brennan, "On the maximum signal-to-noise ratio realizable from several noisy signals," Proc. IRE, vol. 43, p. 1530; October, 1955.

which are monotonically related to the signal-to-noise ratios in the various links. The complex-weighted received signals are summed, then passed into a filter matched to $\xi_m(t)$;[33] the filter output is sampled at $t = T$.

If, on the other hand, the transmission vector $\mathbf{\Gamma}$ has no fixed component, *i.e.*, $\gamma_{2l} = 0$, all l,[15] then (108) reduces to

$$S_m = -\sum_{l=1}^{L} \frac{1}{2N_{0l}} \int_0^T |\zeta_l(t)|^2\, dt + \sum_{l=1}^{L} \frac{\rho_l/4N_{0l}^2}{1 + (\rho_l E_m/N_{0l})} \left| \int_0^T \xi_m^*(t)\zeta_l(t)\, dt \right|^2. \tag{112}$$

Here the first term is again independent of m and may be neglected. The second term is a generalization of the square-law combination of Pierce:[34] before combination, the $\zeta_l(t)$ are passed into filters matched to $\xi_m(t)$,[33] the outputs of the filters being then square-law envelope detected and sampled at $t = T$; the samples are weighted by quantities related to the channel parameters, and the weighted samples are summed. Note that the weights are only equal when $\rho_l = \rho$ and $N_{0l} = N_0$, all l, *i.e.*, when all the links are identical. When the signal-to-noise ratio in the lth link is large—*i.e.*, $\rho_l E_m/N_{0l} \gg 1$—the lth weight is $4/E_m N_{0l}$; at small signal-to-noise ratios, it is $\rho_l/4N_{0l}^2$.

As a final special case, let us consider a simple example in which there is link-to-link correlation of (slow) fading. More precisely, let us consider a dual-diversity case in which $\mathbf{\Gamma}_2 = 0$, $N_{01} = N_{02} = N_0$, and the covariance matrix of $\mathbf{\Gamma}_1$ is

$$\mathbf{G} = \rho \begin{bmatrix} 1 & \lambda \\ \lambda^* & 1 \end{bmatrix}. \tag{113}$$

That is, the complex correlation coefficient between the random fadings γ_{11} and γ_{12} is λ. Then, using (88), (90) and (102) in (105) and (106), we obtain, after some manipulation involving the diagonalization of the quadratic form in (105),

$$S_m = -\frac{1}{2N_0} \sum_{l=1,2} \int_0^T |\zeta_l(t)|^2\, dt + \frac{\rho}{8N_0^2} \left[\frac{(1 + |\lambda|)\,|\alpha_1|^2}{1 + \dfrac{\rho E_m(1 + |\lambda|)}{N_0}} + \frac{(1 - |\lambda|)\,|\alpha_2|^2}{1 + \dfrac{\rho E_m(1 - |\lambda|)}{N_0}} \right], \tag{114}$$

where we have set

$$\alpha_1 = \int_0^T \xi_m^*(t)[\zeta_1(t) + (\lambda/|\lambda|)\zeta_2(t)]\, dt \tag{115}$$

and

$$\alpha_2 = \int_0^T \xi_m^*(t)[\zeta_2(t) - (\lambda^*/|\lambda|)\zeta_1(t)]\, dt. \tag{116}$$

Further, from (107),

$$B_m = -ln \left[1 + \frac{\rho E_m(1 + |\lambda|)}{N_0}\right]\left[1 + \frac{\rho E_m(1 - |\lambda|)}{N_0}\right]. \tag{117}$$

Notice that $\lambda/|\lambda|$ in (115) and (116) is just a phase factor; it is, in fact, a measure of the average phase difference between the signal components of the two received waveforms. The optimum dual-diversity receiver thus first makes an attempt to place the signal components of $\zeta_1(t)$ and $\zeta_2(t)$ in approximate phase coherence by the phase-shifting operations in (114) and (115). The phase-shifted received waveforms are then *coherently* added, (115), and subtracted, (116), and the sum and difference are each passed into a filter matched to $\xi_m(t)$. The squared envelopes of the two filter outputs, sampled at $t = T$, are then combined in the weighted manner indicated in (114).

It is easily seen that (114) and (117) reduce to special cases of (112) and (109), respectively, when $\lambda = 0$, *i.e.*, when the fadings are uncorrelated. When the fadings are identical, *i.e.*, $\lambda = 1$, (114) and (117) readily reduce to the expected result: the two received waveforms should be added at the receiver input and the sum thenceforth treated as a single-diversity signal with a 3-db greater signal-to-noise ratio [*cf.* (112)].

Appendix I

In order to prove that all operations in the optimum receiver may be carried through using only the real parts of complex quantities, we must show that all complex filter matrices are analytic in the sense of (78), and that all vectors involved in various stages of the operations are analytic in the sense of (27) and (28).

The vectors under consideration are, from (42), (45), (68) and (72), $\mathbf{W}(t)$, $\mathbf{U}(t)$, $\hat{\mathbf{V}}(t)$ and $\mathbf{T}(t)$. That $\mathbf{W}(t)$ is analytic follows from the fact that sums and products of analytic functions are analytic; for, since $\xi_m(t)$, $\mathbf{\Gamma}(t)$ and $\mathbf{N}(t)$ were *defined* to be analytic, $\mathbf{Z}(t)$ of (41) and $\mathbf{W}(t)$ of (42) are then also analytic. The other three vectors all appear as filter outputs [see (45), (65), (71)] and are automatically analytic if the filters are analytic, as may be seen through the use of (78) in (74).[21] We now establish this latter condition.

The complex filter matrices we are concerned with are, from (45), (65) and (71), the matrices $\sqrt{\mathbf{Q}(s, t)}$, $\sqrt{\mathbf{P}(s, t)}$ and $\int_0^T \mathbf{P}(s, u)\mathbf{R}_m(u, t)\, du$. Note that all of these are derived from covariance-function matrices: $\mathbf{Q}(s, t)$ is the inverse, in the sense of (10) and (11), of the covariance-function matrix of $\mathbf{N}(t)$; $\mathbf{R}_m(s, t)$ is the covariance-function matrix of $\mathbf{V}(t)$ of (46); and $\mathbf{P}(s, t)$ is the inverse of $\mathbf{I}\delta(s - t) + \mathbf{R}_m(s, t)$.

Now, $\mathbf{N}(t)$ is analytic, so the eigenvectors of its

[33] G. L. Turin, "Error probabilities for binary symmetric ideal reception through nonselective slow fading and noise," Proc. IRE, vol. 46, pp. 1603–1619, Appendix II; September, 1958.

[34] J. N. Pierce, "Theoretical diversity improvement in frequency-shift keying," Proc. IRE, vol. 46, pp. 903–910; May, 1958.

Karhunen-Loève expansion [cf. (3)] must also be analytic.[21] Hence, it follows that $E[\mathbf{N}(s)\mathbf{N}'^*(t)]$, represented in the form of (8) (which has real coefficients), must be analytic in the sense of (78). But from (13) and (14) it is clear that the square root of the inverse of $E[\mathbf{N}(s)\mathbf{N}'^*(t)]$—*i.e.*, $\sqrt{\mathbf{Q}(s, t)}$—has a representation which differs from that of $E[\mathbf{N}(s)\mathbf{N}'^*(t)]$ only in the (real) expansion coefficients, but not in the expansion vectors. Thus $\sqrt{\mathbf{Q}(s, t)}$ must also be analytic.

Since $\sqrt{\mathbf{Q}(s, t)}$ is analytic, $\mathbf{V}(t)$ of (46) is analytic, and by applying the same type of argument as above to $\mathbf{R}_m(s, t)$ and $\mathbf{P}(s, t)$ [see (63)], we may easily conclude that $\mathbf{P}(s, t)$ is analytic, and hence so is $\sqrt{\mathbf{P}(s, t)}$.

Finally, since the representation of the third filter matrix—*i.e.*, $\int_0^T \mathbf{P}(s, u)\mathbf{R}_m(u, t)\, du$—is the same as the first term of (63), except with (real) coefficients $\sigma_k/(1 + \sigma_k)$, it follows that this filter matrix is analytic too.

Appendix II

In order to prove that $\hat{\mathbf{V}}(t)$ is an optimum estimate of $\mathbf{V}(t)$, we have merely to prove that the expansion coefficients of $\hat{\mathbf{V}}(t)$ in (66) are optimum estimates of the expansion coefficients of $\mathbf{V}(t)$ in (49). Note that the observables of the problem are the θ_k, computed from $\mathbf{Y}(t)$ according to (53). Thus if we interpret "optimum" in the maximum-probability sense, we wish to show that the set of coefficients $\{\eta_k\} = \{\sigma_k\theta_k/(1 + \sigma_k)\}$ maximizes the conditional distribution $pr\ [\{\eta_k\}/\{\theta_k\}]$.

Now, the conditional distribution of the $\{\eta_k\}$ may be written as

$$pr\ [\{\eta_k\}/\{\theta_k\}] = \frac{pr\ [\{\eta_k\}]\ pr\ [\{\theta_k\}/\{\eta_k\}]}{pr\ [\{\theta_k\}]}, \tag{118}$$

where the first factor in the numerator, from (55), is

$$pr\ [\{\eta_k\}] = c_F \exp\left[-\sum_k \frac{|\eta_k|^2}{\sigma_k}\right]. \tag{119}$$

In order to find an expression for $pr\ [\{\theta_k\}/\{\eta_k\}]$ in (118), we note from (50), (53) and (84) that θ_k may be written as

$$\theta_k = \eta_k + \epsilon_k, \tag{120}$$

where the ϵ_k are the expansion coefficients of the second term in (84). From the Gaussianness of $\mathbf{N}(t)$ it follows that the ϵ_k are jointly Gaussian, and from (85) it follows that $E[\epsilon_k\epsilon_l^*] = \delta_{kl}$ for *any* orthonormal system of expansion vectors. We may therefore write [*cf.* (36)]

$$\begin{aligned} pr\ [\{\theta_k\}/\{\eta_k\}] &= pr\ [\{\epsilon_k\} = \{\theta_k - \eta_k\}] \\ &= c_1 \exp\left[-\sum_k |\theta_k - \eta_k|^2\right]. \end{aligned} \tag{121}$$

Since we have postulated the signal and noise terms in (84) to be independent, we have further, from (57), $E[|\theta_k|^2] = 1 + \sigma_k$; then

$$pr\ [\{\theta_k\}] = c_2 \exp\left[-\sum_k \frac{|\theta_k|^2}{1 + \sigma_k}\right]. \tag{122}$$

On placing (119), (121) and (122) in (118), we obtain for the *a posteriori* distribution of the η_k:

$$\begin{aligned} &pr\ [\{\eta_k\}/\{\theta_k\}] \\ &= \frac{c_F c_1}{c_2} \exp\left[-\sum_k \frac{|\eta_k - [\sigma_k\theta_k/(1 + \sigma_k)]|^2}{\sigma_k/(1 + \sigma_k)}\right]. \end{aligned} \tag{123}$$

Clearly, the *a posteriori* most probable set of η_k's is that for which the exponent in (123) is zero, *i.e.*, $\{\eta_k\} = \{\sigma_k\theta_k/(1 + \sigma_k)\}$, which was to be proved. Further note that this optimal set of η_k's is the set of conditional means of the η_k; thus the set is optimum in the minimum-variance, as well as the maximum-probability, sense.

Appendix III

As is well known,[35] the covariance-function matrix of a stationary, zero-mean, complex analytic vector process $\mathbf{N}(t)$ has a real part equal to twice the covariance-function matrix of $Re\ \mathbf{N}(t)$, and an imaginary part equal to the Hilbert transform of the real part. Thus, (86) should strictly have been written as

$$E[\mathbf{N}(s)\mathbf{N}'^*(t)] = \begin{bmatrix} N_{01} & & & 0 \\ & N_{02} & & \\ & & \ddots & \\ 0 & & & N_{0L} \end{bmatrix} \left[\delta(s - t) + j\frac{1}{\pi(s - t)}\right]. \tag{124}$$

In order to see the approximation involved in using (86) instead of (124), let us write

$$k(t) = \delta(t) + j\frac{1}{\pi t}, \tag{125}$$

and consider the operation

$$\int_0^T k(s - t)\, x(t)\, dt = \int_{-\infty}^{\infty} k(s - t)x_T(t)\, dt, \tag{126}$$

where $x(t)$ is a complex waveform and

$$x_T(t) = \begin{cases} x(t) & 0 \le t \le T \\ 0 & \text{elsewhere} \end{cases} \tag{127}$$

is a truncation of $x(t)$.

Note that the right-hand side of (126) is the convolution of $k(t)$ with $x_T(t)$. The equivalent operation in the frequency domain is the multiplication of the Fourier transforms of the two functions. Now, the Fourier transform of $k(t)$ is zero for negative frequencies, and 2 for positive frequencies. Thus, if $x_T(t)$ has no negative-frequency components in its Fourier transform, the result of the operation in (126) is merely to multiply the transform of $x_T(t)$, hence $x_T(t)$ itself, by 2. In this case, the effect of $k(t)$ in (126) is precisely the same as that of the operator $2\ \delta(t)$.

Unfortunately, the truncation operation of (127) precludes the complete absence of negative-frequency components in $x_T(t)$. But if $x(t)$ is complex-analytic, as are

[35] See, *e.g.*, M. Zakai, "Second-order properties of pre-envelope and envelope processes," IRE Trans. on Information Theory, vol. IT-6, pp. 556–559; December, 1960. Note, however, that for a process with nonzero mean this statement is true only within an additive constant equal to the square of the mean of the $Re\ \mathbf{N}(t)$ process.

all the waveforms in this paper, then it has no negative-frequency components; if, further, only a small fraction of the power (or energy) in $x(t)$ lies below roughly $1/T$ cps in frequency (this includes most applications of interest here), then the truncation will produce no significant negative-frequency components in $x_T(t)$, and the previous argument concerning the equivalence of $k(t)$ and $2\ \delta(t)$ in (126) holds to a very good approximation. To this approximation, whenever (124) is used as an integral operator in the manner of (126)—and this is uniformly its use in this paper—(86) may be used in its stead.

Appendix IV

On placing (103) into (51), we obtain

$$\mathbf{M}\xi_m(s) \int_0^T \xi_m(t)\mathbf{\Psi}(t)\, dt = \sigma\mathbf{\Psi}(s), \qquad 0 \leq s \leq T. \tag{128}$$

Solutions of this are clearly of the form $\mathbf{\Psi}_k(t) = \mathbf{F}_k\xi_m(t)$, where $\mathbf{F}_k$ is a time-invariant vector which, from (128), must satisfy the set of algebraic equations

$$2E_m\mathbf{MF} = \sigma\mathbf{F}. \tag{129}$$

Since the σ_k are the eigenvalues of (129), $(1 + \sigma_k)$ are the eigenvalues of

$$(\mathbf{I} + 2E_m\mathbf{M})\mathbf{F} = \lambda\mathbf{F}, \tag{130}$$

whence, by a well-known result from the theory of linear equations,

$$|\ \mathbf{I} + 2E_m\mathbf{M}\ | = \prod_k (1 + \sigma_k). \tag{131}$$

Insertion of (131) into (60) leads immediately to (107).

VI. Acknowledgment

It is a pleasure for the author to acknowledge the enlightening discussions he had with Dr. D. R. Anderson and Dr. A. V. Balakrishnan on some of the mathematical theory upon which this work is based.

A New Derivation of the Entropy Expressions*

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Summary—**In the discrete case, the Shannon expression for entropy is obtained as a line integral in probability space. The integrand is the "information density vector" ($\log p_1, \log p_2, \cdots, \log p_n$). In the continuous case, the continuous analog of information density is integrated to obtain the entropy expression for continuous probability distributions.**

CONSIDER the integral

$$\begin{aligned}\int_a^b \log \frac{x}{1-x}\, dx &= \int_a^b \log x\, dx - \int_a^b \log(1-x)\, dx \\ &= [x \log x - x]_a^b + [u \log u - u]_{1-a}^{1-b} \\ &= [b \log b + (1-b) \log (1-b)] \\ &\quad - [a \log a + (1-a)\log(1-a)] \\ &= H(b, 1-b) - H(a, 1-a),\end{aligned} \tag{1}$$

where $H(x, 1-x)$ is Shannon's entropy function.

If an experiment has two possible outcomes, which are assigned *a priori* probabilities a and $1 - a$, but, after receipt of further information, are assigned the *a posteriori* probabilities b and $1 - b$, the net change in information (*i.e.*, the quantity of additional information received) is measured by (1). This suggests the definition $D(x, 1 - x) = \log [x/(1 - x)]$ as the *information density* for an experiment having two possible outcomes, with probabilities x and $1 - x$. Specifically, the information density $D(x, 1 - x)$ has the property that integration from $x = a$ to $x = b$ yields the net change in information when the probability assigned to x is changed from a to b.

If p and q are probabilities, $p + q = 1$, then $D(p, q) = \log (p/q)$. This function frequently occurs as a criterion function in statistical decision theory. For those interested in the axiomatic approach, it suffices to seek an "information density function" $D(p, q)$ which satisfies the single axiom

$$k\ D(p, q) = D\left(\frac{p^k}{p^k + q^k}, \frac{q^k}{p^k + q^k}\right). \tag{2}$$

Formally, this may be treated as follows:

Theorem: The only function $D(p, q)$ which satisfies (2) for all p with $0 < p < 1$ and all real k is $D(p, q) = c \log (p/q)$ (where the constant c can be considered a change of logarithmic base).

Proof: With $k = 0$ in (2), it is seen that $D(\frac{1}{2}, \frac{1}{2}) = 0$;

* Received by the PGIT, September 30, 1960. This paper presents the results of one phase of research carried out at the Jet Propulsion Laboratory, California Institute of Technology, under Contract No. NASw-6, sponsored by the National Aeronautics and Space Administration.

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and with $k = -1$, it is seen that $D(p, q) = -D(q, p)$. If $D(p, q)$ is not identically zero (a degenerate case which corresponds to $c = 0$), then there is a value $p = \alpha \neq \frac{1}{2}$ such that $D(\alpha, 1 - \alpha) \neq 0$. For any p with $0 < p < 1$, there is a real number k such that $\alpha = p^k/(p^k + q^k)$. Specifically,

$$\alpha = \frac{1}{1 + \left(\frac{1}{p} - 1\right)^k} \quad \text{gives} \quad \left(\frac{q}{p}\right)^k = \frac{1}{\alpha} - 1,$$

and

$$k = \log \frac{1 - \alpha}{\alpha} \Big/ \log \frac{q}{p}.$$

Using (2), we find that

$$D(p, q) = \frac{1}{k} D(\alpha, 1 - \alpha) = \left(\log \frac{p}{q}\right)\left(D(\alpha, 1 - \alpha)/\log \frac{\alpha}{1 - \alpha}\right).$$

If we let

$$c = D(\alpha, 1 - \alpha)/\log \frac{\alpha}{1 - \alpha},$$

which is legitimate since $\alpha \neq \frac{1}{2}$ and $0 < \alpha < 1$, the theorem follows as stated.

The principal "justification" for the axiom (2) is that it leads to the desired theory via an interesting route. However, there is a simpler-looking formulation of (2) which is fully equivalent, obtained by defining information density for *odds* rather than strictly for probabilities (*i.e.*, normalized odds). Specifically, if the odds change from $F : G$ to $F^k : G^k$, the information density is multiplied by k:

$$k\, D(F : G) = D(F^k : G^k).$$

In order to generalize (1) to experiments with n possible outcomes, we must perform a line integration from the *a priori* vector of probabilities $r = (r_1, r_2, \cdots, r_n)$ to the *a posteriori* vector of probabilities $s = (s_1, s_2, \cdots, s_n)$. The integrand is now the vector $(\log x_1, \log x_2, \cdots, \log x_n)$, integrated with respect to the vector $(dx_1, dx_2, \cdots, dx_n)$. In this way,

$$\int_r^s (\log x_1, \log x_2, \cdots, \log x_n)\cdot(dx_1, dx_2, \cdots, dx_n) = \sum s_i \log s_i - \sum r_i \log r_i = H(s) - H(r) \tag{3}$$

as desired.

Indeed, for the special case $n = 2$, (3) reduces to

$$\begin{aligned}\int_{a,1-a}^{b,1-b} (\log x_1, \log x_2)\cdot(dx_1, dx_2) &= \int_a^b [\log x\, dx + \log(1 - x)\, d(1 - x)] \\ &= \int_a^b [\log x\, dx - \log(1 - x)\, dx] = \int_a^b \log \frac{x}{1 - x}\, dx,\end{aligned}$$

which is the original integral (1).

Essentially, (3) expresses the notion that when the odds on n possible outcomes are $x_1 \colon x_2 \colon \cdots \colon x_n$, then the local information flux (*i.e.*, density) is represented by $(\log x_1, \log x_2, \cdots, \log x_n)$, there being a separate component for each of the possible outcomes. When this local behavior is accumulated (*i.e.*, integrated) from initial point r to terminal point s in probability space, the result is the total difference in information between r and s.

For continuous information, a very proper passage to the continuous case of (3) gives the desired entropy expression. (Previous attempts based on the entropy rather than the information density have run into serious obstacles.) Specifically, letting x_i be replaced by the continuous probability distribution $x(t)$, so that $\sum (\log x_i)\cdot(dx_i)$ is replaced by $\int (\log x(t))(dx(t))$, the analog of (3) is

$$\begin{aligned}\int_{-\infty}^{\infty} \left[\int_{x_0(t)}^{x_1(t)} \log x(t)\, dx(t)\right] dt &= \int_{-\infty}^{\infty} [u \log u - u]_{x_0(t)}^{x_1(t)}\, dt \\ &= \int_{-\infty}^{\infty} x_1(t) \log x_1(t)\, dt - \int_{-\infty}^{\infty} x_0(t) \log x_0(t)\, dt \\ &= H(x_1(t)) - H(x_0(t)),\end{aligned} \tag{4}$$

where again $H(x(t))$ is the expression for entropy recommended by Shannon.

The Use of Group Codes in Error Detection and Message Retransmission*

W. R. COWELL†

Summary—The paper considers group codes whose function is split between error correction and error detection with retransmission. For a given code, the minimum error probability is obtained when retransmission occurs whenever an error is detected. An estimate of the redundancy added by retransmission is given and the behavior of retransmission channels as the length of the code words increases is studied. Most of the analysis is for the binary symmetric channel, although some of the results apply to more general channels.

Introduction

Much of the recent work in coding theory for binary digital channels has involved a search for codes which have good error detecting and correcting properties and yet may be instrumented easily. Many communication links on which such codes would be used permit the transfer of information in both directions so that it is possible, just as in human conversation, for the receiver to request the retransmission of messages or parts of messages in which errors are detected. It is the purpose of this paper to consider such an error control plan in which a group code is used as the error detector. Much of the analysis is carried out for the binary symmetric channel; we will note which results hold for more general channels.

I. The Decoder

We will consider first a decoder which makes both correction decisions and retransmission decisions depending on the received word. Suppose that the words of length n of a group code X are the input to a binary symmetric channel with transition probability p where $p \leq \frac{1}{2}$. At the receiver is a decomposition into cosets[1] relative to X of the group of all binary sequences of length n under componentwise modulo 2 addition. Set A of coset "leaders" is selected so that A contains exactly one member of each coset and includes the 0 sequence as the leader of X. Let S be a subset of A which contains 0. The decoder operates as follows: A received sequence y is expressed (uniquely) as $y = a + x$ where a is in A and x is in X. If a is in S, y is decoded to x. If a is not in S, the transmitter is instructed, via a reverse channel, to retransmit the code word. We will assume that the reverse channel operates without error, that retransmissions are independent, and that a given word is retransmitted until a word of form $s + x$ for s in S is received.

If we define an error pattern of length n as a sequence of binary digits in which 0 represents a correct digit and 1 represents an error, then we observe that our decoder corrects those error patterns which are words of S and requests retransmission when the received word lies in a coset whose leader is not in S. If S is the zero sequence alone, the retransmission occurs whenever the received word is not a code word. If the weight of a sequence is the number of 1's in the sequence, then the case where each element of A has minimal weight in its coset and $S = A$ gives the "maximum likelihood detector" studied by Slepian [4] and others.

Let $w(x)$ denote the weight of the sequence x, and $d(x, y)$ be the Hamming distance [3] from x to y. Note that $d(x, y) = w(x + y)$. We define $\eta(x) = p^{w(x)}q^{n-w(x)}$ where $q = 1 - p$. Then the probability that y is observed at the receiver when x is transmitted is $\eta(y + x)$.

Let θ_x be the probability that x is retransmitted following a transmission of x. Thus, $1 - \theta_x$ is the probability that when x is transmitted we observe at the receiver a sequence of form $s + x'$ where s is in S and x' is in X. This probability is

$$1 - \theta_x = \sum_{s \in S} \sum_{x' \in X} \eta(s + x' + x),$$

and is clearly independent of x; we may write

$$1 - \theta = \sum_{s \in S} \sum_{x \in X} \eta(s + x).$$

Let D_x be the probability that if x is transmitted then the sequence observed at the receiver decodes into x. This is simply the probability that the observed sequence is of the form $s + x$ where s is in S and, therefore,

$$D_x = \sum_{s \in S} \eta(s + x + x) = \sum_{s \in S} \eta(s).$$

Since this is independent of x, we shall write $D_x = D$. Now the probability of decoding into x after exactly r retransmissions, given that x was transmitted, is $\theta^r D$, and so the probability of ultimately decoding into x given that x was transmitted is

$$\sum_{i=0}^{\infty} \theta^i D.$$

This is clearly independent of x and we may write

$$J = \sum_{i=0}^{\infty} \theta^i D = \frac{D}{1 - \theta}$$

as the probability that a word is decoded correctly. Henceforth, we will use the verb "to decode" in the sense "to decode ultimately, possibly after retransmissions have taken place."

Two special cases are worthy of note. If $S = A$, then $\theta = 0$ and $J = D = \sum_{a \in A} \eta(a)$, the sum over the coset leaders. If S is the 0 word alone, then $1 - \theta = \sum_{x \in X} \eta(x)$ and $D = q^n$ so $J = q^n/(1 - \theta)$.

* Received by the PGIT, October 28, 1960.

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[1] See [4] for a discussion of the algebraic properties of group codes.

heorem 1

Given a group code X, let J^* be the probability that word is decoded correctly when S is the zero word and t J be the corresponding probability for any other hoice of S. Then $J^* \geq J$.

Proof: The weight function satisfies the triangle inquality: $w(x+y) \leq w(x) + w(y)$ for all sequences x nd y. We recall that $p \leq \frac{1}{2}$. Therefore,

$$\eta(x+y) = p^{w(x+y)} q^{n-w(x+y)} \geq p^{w(x)+w(y)} q^{n-w(x)-w(y)}$$

$$= \frac{1}{q^n} p^{w(x)} q^{n-w(x)} p^{w(y)} q^{n-w(y)} = \frac{\eta(x)\cdot\eta(y)}{q^n}.$$

Let θ be the probability of retransmission for the hoice of S which contains more than the element 0 and * be the probability of retransmission when S is the ero word. Then,

$$1 - \theta = \sum_{s\varepsilon S} \sum_{x\varepsilon X} \eta(s+x) \geq \sum_{s\varepsilon S} \sum_{x\varepsilon X} \frac{\eta(s)\cdot\eta(x)}{q^n}$$

$$= \frac{1-\theta^*}{q^n} \sum_{s\varepsilon S} \eta(s) = \frac{1-\theta^*}{q^n}\cdot D.$$

Thus,

$$J^* = \frac{q^n}{1-\theta^*} \geq \frac{D}{1-\theta} = J,$$

hich was to be shown.

Therefore, the probability of correct decoding is greatest r the case when retransmission occurs whenever a word ifferent from a code word is received. This is intuitively lausible when we realize that this is the case when the robability of retransmission is maximal and, hence, the edundancy introduced by retransmission is greatest. We roceed next to obtain a more precise formulation of this dded redundancy.

II. Code Efficiency

Suppose the code words of a group code X have m nformation places and $f = n - m$ parity check places. Define the *efficiency* of X used as a corrector-detector as he ratio of m to the mean number of digits transmitted ntil a word is decoded at the receiver. Suppose further hat $L \geq 0$ digits are "lost" or "wasted" whenever a etransmission takes place. We will think of these lost igits as adding to the total number of digits transmitted ntil a given word is decoded. In a real system that equires a certain time to "reset" or "turn around" reparatory to retransmission the digits that would have een transmitted during this reset time would be rearded as lost as would digits used to re-establish synhronization, digits thrown away because of an intereaved transmission pattern, etc. In determining L, we ill *not* include the digits of the retransmitted code ord itself.

heorem 2

If a group code of length n has m information places nd is used with a corrector-detector such that the probability of retransmission is θ, and if L digits are lost on each retransmission, the efficiency of the code is

$$e = \frac{m(1-\theta)}{n + L\theta}.$$

Proof: If exactly i retransmissions occur before a given word is decoded; *i.e.*, the word is decoded on the $(i+1)$ *st* transmission, then the number of digits transmitted is $(i+1)n + iL = n + i(n+L)$. The probability of exactly i retransmissions before decoding is $\theta^i(1-\theta)$ and so the expected number of digits transmitted until decoding is

$$\sum_{i=0}^{\infty} \theta^i(1-\theta)[n + i(n+L)]$$

$$= n(1-\theta)\sum_{i=0}^{\infty}\theta^i + (n+L)(1-\theta)\sum_{i=0}^{\infty} i\theta^i$$

$$= n + (n+L)\frac{\theta}{1-\theta}.$$

Therefore,

$$e = \frac{m}{n + (n+L)\dfrac{\theta}{1-\theta}} = \frac{m(1-\theta)}{n+L\theta} \quad \text{as required.}$$

Corollary

Assume $\theta > 0$. Then a necessary and sufficient condition that the efficiency of the corrector-detector of Theorem 2 be at least as great as the efficiency of a group code of length $n + \Delta$ with m information places which is used as a corrector only, is

$$n + L \leq \Delta\cdot\frac{1-\theta}{\theta}.$$

Proof: The assertion may be stated as follows:

$$n + L \leq \Delta\frac{1-\theta}{\theta}$$

if and only if

$$\frac{m(1-\theta)}{n+L\theta} \geq \frac{m}{n+\Delta}.$$

This is easily obtained by simple manipulation of the inequalities.

It may be remarked that Theorem 2 and its corollary do not make use of the binary symmetric property and, hence, could be stated so as to apply to more general channels.

As a numerical example, let us take the group code with $n = 8$, $m = 4$ which is listed by Slepian [4] as the best corrector with these parameters. Take $L = 100$ and use the code as a detector only; *i.e.*, S is the zero word. For several values of p, Table I lists θ, e, and the smallest positive integer Δ for which the inequality of the corollary is satisfied. The last column is the probability that a word is decoded in error when the code is used for detection and retransmission only.

TABLE I

p	θ	e	Δ	$1 - J$
10^{-1}	5.67×10^{-1}	2.67×10^{-2}	142	5.2×10^{-3}
10^{-2}	7.73×10^{-2}	2.35×10^{-1}	10	3.1×10^{-6}
10^{-3}	7.97×10^{-3}	4.51×10^{-1}	1	$< 10^{-7}$
10^{-4}	8×10^{-4}	4.95×10^{-1}	1	$< 10^{-7}$
10^{-5}	8×10^{-5}	4.99×10^{-1}	1	$< 10^{-7}$

III. Limiting Behavior as Word Length Increases

In this section, we will be concerned with the case where S is the zero word and thus retransmission occurs whenever a word that is not a code word is observed at the receiver. We will call the ratio $r = 1 - m/n$ the *code redundancy*. It should be noted that $r < 1 - e$ because of the decrease in efficiency caused by retransmission.

By an *r-sequence of group codes* we mean a sequence of group codes of lengths b, $2b$, $3b$, $\cdots$ which have a, $2a$, $3a$, $\cdots$ check digits respectively where a and b are fixed, $a \neq 0$, $b > a$ and $r = a/b$ is in lowest terms. Thus, the code redundancy remains fixed while the lengths increase. The code of the sequence of length bc will be referred to as the cth code of the sequence and designated by X_c. Notationally $n = bc$ and $m = (b - a)c$.

Now, for any r-sequence, let θ_c be the probability of retransmission for the cth code. Then,

$$1 - \theta_c = \sum_{x \in X_c} p^{w(x)} q^{n-w(x)} \leq \sum_{i=0}^{m} \binom{m}{i} p^i q^{n-i},$$

where we have replaced the check digits of each code word with 0's in order to obtain an upper bound. Hence,

$$1 - \theta_c \leq q^{n-m} \sum_{i=0}^{m} \binom{m}{i} p^i q^{m-i} = q^{n-m} = q^{ac}.$$

Therefore,

$$\lim_{c\to\infty} (1 - \theta_c) = 0 \quad \text{so} \quad \lim_{c\to\infty} \theta_c = 1.$$

From Theorem 2, the efficiency of the cth code is

$$e_c = \frac{(1 - r)(1 - \theta_c)}{1 + \dfrac{L_c\theta_c}{bc}},$$

and consequently $\lim_{c\to\infty} e_c = 0$.

Thus, as the code words increase in length, the probability of detecting errors and retransmitting increases toward 1 and the efficiency decreases toward 0. It is reasonable to ask whether there exist r-sequences such that the probability of correct decoding approaches 1 with increasing code length. This question is answered by the following theorem. (All logarithms are to the base 2.)

Theorem 3

If $r > -\log q$, there exists an r-sequence such that $\lim_{c\to\infty} J_c = 1$ where J_c is the probability that a word is decoded correctly for the cth code. Moreover, if the input is random and $H(X_c \mid X_c')$ is the equivocation *per word* for the cth code then $\lim_{c\to\infty} H(X_c \mid X_c') = 0$.

Proof: A group code of length n with m information places and $f = ac$ check places is uniquely defined by a parity check matrix of 0's and 1's with m rows and f columns. There are 2^{mf} such matrices and 2^{mf} codes. For many purposes these codes are not all distinct, but it suits our purposes to consider them as different here since we wish to calculate the average of $1 - \theta$ over this set of codes.

We need the following combinatorial result whose proof may be found elsewhere:[2] Suppose that some sequence of m information digits which contains 1 in at least one place is given. If we write the corresponding sequence of check digits for each of the 2^{mf} possible group codes, we find that each of the 2^f possible sequences of length f occurs exactly $2^{(m-1)f}$ times as a sequence of check digits.

Now, sum $1 - \theta$ over all possible group codes with m information places and f check places. First, sum $p^{w(x)}q^{n-w(x)}$ over all code words that have some fixed information sequence with 1 in at least one place. Let the weight of the information places be $i \neq 0$.

$$\sum_{j=0}^{f} \binom{f}{j} 2^{(m-1)f} p^{i+j} q^{m+f-i-j}$$

$$= 2^{(m-1)f} p^i q^{m-i} \sum_{j=0}^{f} \binom{f}{j} p^j q^{f-j} = 2^{(m-1)f} p^i q^{m-i}.$$

When the information places are all 0, the check places are 0 for every group code, and therefore the sum of $p^{w(x)}q^{n-w(x)} = q^n$ over these code words for all codes is $2^{mf}q^n$.

Having obtained the sum for each sequence of information digits, we take the sum of these sums; *i.e.*, the sum over the sequences of information digits:

$$2^{mf}q^n + \sum_{i=1}^{m} \binom{m}{i} 2^{(m-1)f} p^i q^{m-i}$$

$$= 2^{mf}q^n + 2^{(m-1)f}\left[\left(\sum_{i=0}^{m} \binom{m}{i} p^i q^{m-i}\right) - q^m\right]$$

$$= 2^{mf}[q^n + 2^{-f}(1 - q^m)].$$

Hence, the average of $1 - \theta$ over the possible group codes is

$$q^n + 2^{-f}(1 - q^m).$$

Now let $r > -\log q$ and construct an r-sequence as follows: For each c, select a code for which $1 - \theta_c$ is no more than the average calculated above. Such a code always exists, of course. Then, for this r-sequence, consider

$$n(1 - J_c) = n - \frac{nq^n}{1 - \theta_c} \leq n - \frac{nq^n}{q^n + 2^{-f}(1 - q^m)}$$

$$= \frac{n2^{-f}(1 - q^m)}{q^n + 2^{-f}(1 - q^m)}$$

$$= \frac{1}{\dfrac{1}{n} + \dfrac{(2^r q)^n}{n} \cdot \dfrac{1}{1 - q^m}}.$$

[2] See [2], ch. 7.

Evidently

$$\lim_{c\to\infty} \frac{1}{n} = \lim_{c\to\infty} \frac{1}{bc} = 0$$

and

$$\lim_{c\to\infty} \frac{1}{1 - q^m} = \lim_{c\to\infty} \frac{1}{1 - q^{(b-a)c}} = 1,$$

and the condition $r > -\log q$ guarantees that $2^r q > 1$; therefore,

$$\lim_{c\to\infty} \frac{(2^r q)^n}{n} = \lim_{c\to\infty} \frac{(2^r q)^{bc}}{bc} = \infty.$$

Therefore, $\lim_{c\to\infty} n(1 - J_c) = 0$ and the first assertion of the theorem is immediate.

To prove the second statement let us recall that the probability that a word y is observed at the receiver when x is transmitted is $\eta(y + x)$. Therefore, the probability of decoding to the code word x' when x is transmitted is

$$Pr(x' \mid x) = \sum_{i=0}^{\infty} \theta^i \eta(x' + x) = \frac{\eta(x' + x)}{1 - \theta} = Pr(x \mid x').$$

For any code X, $H(X \mid X')$ is the expected value of

$$Q(x') = -\sum_{x \in X} Pr(x \mid x') \log Pr(x \mid x')$$

relative to the x'. By the randomness condition, the expected value is the unweighted average of $Q(x')$ over the code words. By the group property, $Q(x')$ is independent of x' so that

$$H(X \mid X') = -\sum_{x \in X} \frac{\eta(x)}{1 - \theta} \log \frac{\eta(x)}{1 - \theta}.$$

Using the definition of $\eta(x)$, this becomes

$$H(X \mid X') = -\sum_{x \in X} \frac{\eta(x)}{1 - \theta}$$

$$\cdot [w(x) \log p + (n - w(x)) \log q - \log (1 - \theta)]$$

$$= \frac{\log q - \log p}{1 - \theta} \sum_{x \in X} w(x)\eta(x) - \log \frac{q^n}{1 - \theta} \sum_{x \in X} \frac{\eta(x)}{1 - \theta}$$

$$= (\log q - \log p)G - \log J,$$

where

$$G = \frac{1}{1 - \theta} \sum_{x \in X} w(x)\eta(x).$$

We note parenthetically that G is the expected number of digits in error per received word. The property of G of importance in the present context is that G is dominated by $n(1 - J)$ for

$$G = \frac{1}{1 - \theta} \sum_{\substack{x \neq 0 \\ x \in X}} w(x)\eta(x) \leq \frac{n}{1 - \theta} \sum_{x \neq 0} \eta(x)$$

$$= \frac{n}{1 - \theta} (1 - \theta - q^n) = n - \frac{nq^n}{1 - \theta} = n(1 - J).$$

Hence, $\lim_{c\to\infty} G_c = 0$ for the r-sequence constructed above.

Thus, from the above formula for $H(X \mid X')$,

$$\lim_{c\to\infty} H(X_c \mid X_c') = 0.$$

This completes the proof.

IV. Some Unsolved Problems

We mention finally several open questions which may be worthy of investigation.

1) For given n and m, what is the best retransmission code (in the sense of maximizing J)? This question is probably very difficult to answer in general but is of practical significance for small m and n. The best correction code is not necessarily the best retransmission code. For example, a certain code with $n = 7$, $m = 3$ mentioned by Slepian[3] has a higher probability of correct decoding when used as a retransmission code than does the "best" group code (in the correction sense) when the latter is used as a retransmission code.

2) Can we choose an r-sequence of codes X_c and define a set of coset leaders S_c for each c so that $\lim_{c\to\infty} \theta_c$ is neither 1 nor 0 and yet $\lim_{c\to\infty} J_c = 1$?

3) In Theorem 3 is $-\log q$ the best bound on r? We may remark that the work of Elias [1] together with our Theorem 1 guarantees that when $r > -p \log p - q \log q$ there is an r sequence such that $J \to 1$. However, Theorem 3 is stronger, not only because of the result on equivocation but also because

$$-\log q < -p \log p - q \log q.$$

4) For a given r-sequence define

$$R(c) = e_c - \frac{1}{n} H(X_c \mid X_c')$$

as the effective rate per digit of transmitting information for the cth code. It is easy to show that $\lim_{c\to\infty} R(c) = 0$. Can one find the maxima of $R(c)$ with respect to c? If so, this could lead to a definition of optimum code length for the given r-sequence.

5) The dominating practical question, of course, is how to combine the assurances of Theorem 3 with the estimate of efficiency of Theorem 2 and construct codes and coding equipment so that $J > 1 - \epsilon$ and yet the efficiency, complexity of the instruments, and cost are tolerable when ϵ is reasonably small. Any attempt to answer this question for a particular system introduces variables which we have not considered here. However, Theorem 1 indicates that retransmission as a method of error control deserves further practical attention.

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On the Factorization of Rational Matrices*

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Summary—Many problems in electrical engineering, such as the synthesis of linear n ports and the detection and filtration of multivariable systems corrupted by stationary additive noise, depend for their successful solution upon the factorization of a matrix-valued function of a complex variable p.

This paper presents several algorithms for affecting such decompositions for the class of rational matrices $G(p)$, *i.e.*, matrices whose entries are ratios of polynomials in p. The methods employed are elementary in nature and center around the Smith canonic form of a polynomial matrix. Several nontrivial examples are worked out in detail to illustrate the theory.

I. Introduction

It is well known [1]–[3] that many problems involving the detection and filtration of multivariable systems contaminated by stationary additive noise can be reduced to the study of a matrix Wiener-Hopf integral equation of the type

$$\int_0^{\infty} K(t-\tau)\mathbf{W}(\tau)\,d\tau = \mathbf{e}(t), \qquad t > 0, \tag{1}$$

where $K(t)$ is the covariance matrix of the noise, $\mathbf{e}(t)$ is a deterministic column-vector function prescribed in advance by the known datum, and $\mathbf{W}(\tau)$ is the unknown column vector of filter weighting functions $W_1(\tau)$, $W_2(\tau)$, $\cdots$, $W_n(\tau)$. In most practical cases, the noise possesses a rational absolutely continuous spectral density matrix:

$$K(t) = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} G(p)e^{pt}\,dp, \qquad j = \sqrt{-1}, \tag{2}$$

where

$$G(p) = \int_{-\infty}^{+\infty} K(t)e^{-pt}\,dt \tag{3}$$

is $n \times n$ and has rational entries. Moreover [4],

1) $\bar{G}(p) = G(\bar{p})$.
2) $G'(-p) = G(p)$.
3) $\mathbf{b}^*G(j\omega)\mathbf{b} \geq 0$ for every n-vector $\mathbf{b}$ and every real finite ω. For short, $G(j\omega) \geq 0$.

To solve (1) by the Wiener-Hopf technique it suffices to exhibit a factorization of $G(p)$ of the form (A' denotes the transpose of the matrix A)

$$G(p) = H'(-p)H(p) \tag{4}$$

* Received by the PGIT, October 18, 1960. The work reported herein was sponsored by the AF Cambridge Res. Ctr., L. G. Hanscom Field, Bedford, Mass., under Contract No. AF–19(604)–4143, and presented as Res. Rept. PIBMRI 855–60.

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with the following properties [11], [2]:

1) $H(p)$ is rational and analytic together with its inverse $H^{-1}(p)$ in a right half-plane $Re\ p > -\mu$, $\mu > 0$.
2) $H(p)$ is real; *i.e.*, $\bar{H}(p) = H(\bar{p})$.

The object of this paper is to describe a specific algorithm for affecting such decompositions for the class of rational matrices and to consider some related questions.

II. Preliminary Notation and Definitions

Let A be an arbitrary matrix. Then A', $\bar{A}$, A^*, A^{-1} and $|A|$ denote the transpose, the complex conjugate, the adjoint ($\bar{A}'$), the inverse and the determinant of A, respectively.

A diagonal matrix A with diagonal elements μ_1, μ_2, $\cdots$, μ_n is written as $A = \text{diag}\,[\mu_1, \mu_2, \cdots, \mu_n]$. Column vectors are represented by $\mathbf{x}$, $\mathbf{y}$, etc., or in the alternative fashion $\mathbf{x} = (x_1, x_2, \cdots, x_n)'$ whenever it is desirable to indicate the components explicitely; 1_n, $\mathbf{O}_n$ and $O_{n,m}$ are, in the same order, the $n \times n$ identity matrix, the n-dimensional zero vector and the $n \times m$ zero matrix.

A matrix $A(p)$ is polynomial if each of its entries is a polynomial in p. $A(p)$ is rational if each of its elements is rational in p; *i.e.*,

$$(A)_{rk} = \frac{b_{rk}(p)}{g_{rk}(p)},$$

$f_{rk}(p)$ and $g_{rk}(p)$ being polynomials.

$A(p)$ is said to be real if $\bar{A}(p) = A(\bar{p})$. In particular, $\bar{A}(j\omega) = A(-j\omega)$ for all real ω.

The non-negative integer $r(A)$ is the normal rank of the rational matrix $A(p)$ if 1) there exists at least one subminor of order r which does not vanish identically, and 2) all minors of order greater than r vanish identically. Clearly the normal rank of a rational matrix can decrease at most on a finite set of points in the p plane.

A nonsquare matrix does not possess an inverse in the ordinary sense. However, it may have either a right or left inverse. Thus if A is $m \times n$, A possesses a right inverse A^{-1}, such that $AA^{-1} = 1_m$ if and only if $m \leq n$ and $r(A) = m$.

An elementary polynomial matrix is a polynomial matrix possessing either a right or left polynomial inverse. A square matrix $A(p)$ is elementary if and only if its determinant is a constant independent of p.

$A(p)$ is analytic in a region of the p plane if all its entries are analytic in this region.

The point p_0 is a pole of $A(p)$ if some element of $A(p)$ has a pole at $p = p_0$.

If p_0 is a pole of the rational matrix $A(p)$, each element of A may be expanded in partial fractions and after

collecting all those terms having poles at p_0 there is obtained for $p_0 \neq \infty$,

$$A(p) = (p - p_0)^{-k}A_k + (p - p_0)^{-k+1}A_{k-1} + \cdots + (p - p_0)^{-1}A_1 + A_0(p), \qquad (5)$$

where $A_0(p_0)$ is finite, $A_k \neq O$ and the A_i, $1 \leq i \leq k$, are constant matrices. If $p_0 = \infty$, $(p - p_0)^{-i}$ is replaced by p^i, $1 \leq i \leq k$. All of $A_0(p)$, $A_1, \cdots, A_k$ are uniquely defined by their construction from $A(p)$.

Def. 1: If $A(p)$ is given by (5), then k is the order of the pole of $A(p)$ at $p = p_0$.

Def. 2: A rational matrix $A(p)$ is said to be paraconjugate hermetian if $A^*(p) = A(-\bar{p})$. Hence, on the real-frequency axis $p = j\omega$, $A^*(j\omega) = A(j\omega)$ and $A(j\omega)$ is hermetian in the ordinary sense. For real $A(p)$, $A^*(-\bar{p}) = A'(-p)$ and the paraconjugate condition simplifies to $A'(-p) = A(p)$. A real paraconjugate hermetian matrix is called para-hermetian.

Def. 3: A rational $m \times n$ matrix $A(p)$ is said to be paraconjugate unitary if either $A^*(-\bar{p})A(p) = 1_n$, or $A(p)A^*(-\bar{p}) = 1_m$, or both. On the real-frequency axis $p = j\omega$, $A^*(-\bar{p}) = A^*(j\omega)$ and $A(j\omega)$ is unitary in the usual sense. For real $A(p)$ the paraconjugate unitary condition simplifies to $A'(-p)A(p) = 1_n$ or $A(p)A'(-p) = 1_m$. A real paraconjugate unitary matrix is para-unitary.

It is most convenient for typographical reasons to let

$$A_*(p) \equiv A^*(-\bar{p}).$$

This notation is used throughout the remainder of the paper. Observe that $A_{**}(p) = A(p)$ and $(AB)_* = B_*A_*$.

A scalar function $f(p)$ satisfying $\bar{f}(-\bar{p}) = f(p)$ is called paraconjugate. If $f(p)$ is real and paraconjugate, it is actually even.

Def. 4: A paraconjugate unitary matrix $A(p)$ is said to be regular if it is analytic in the right-half plane $Re\ p > 0$.

The structure of rational matrices is the subject of the classic Smith-McMillan lemma.

Smith-McMillan Lemma [5], [8]: Let $G(p)$ be an $m \times n$ rational matrix of normal rank r. Then there exist two elementary polynomial matrices $C(p)$ and $F(p)$ of orders $m \times r$ and $r \times n$, respectively, such that

$$G(p) = C(p)\ \mathrm{diag}\left[\frac{e_1(p)}{\psi_1(p)}, \frac{e_2(p)}{\psi_2(p)}, \cdots, \frac{e_r(p)}{\psi_r(p)}\right]F(p) \qquad (6)$$

$$= CDF,$$

where

a) $e_k(p)$ and $\psi_k(p)$ are relatively prime polynomials with unit leading coefficients, $1 \leq k \leq r$;
b) Each $e_k(p)$ divides $e_{k+1}(p)$, $1 \leq k \leq r - 1$, and each $\psi_l(p)$ is a factor of $\psi_{l-1}(p)$, $2 \leq l \leq r$;
c) The diagonal matrix $D(p)$ appearing in (6) is, subject to a) and b), uniquely determined by $G(p)$. It is, in fact, canonic;
d) If $G(p)$ is real, the e's, ψ's, $C(p)$ and $F(p)$ may also be chosen real;
e) The finite point $p = p_0$ is a pole of $G(p)$ of order k if and only if it is a zero of $\psi_1(p)$ of order k;
f) The order of $p = \infty$ as a pole of $G(p)$ is the same as the order of $z = 0$ as a pole of $\hat{G}(z) \equiv G(1/z)$.

A rational matrix is said to be canonic if it is square, nonsingular and diagonal with the properties a) and b) listed in the Smith-McMillan lemma. The rational functions e_1/ψ_1, e_2/ψ_2, $\cdots$, e_r/ψ_r are generalized "invariant factors" of $G(p)$. For the sake of brevity, the above lemma is referred to as the S.M. lemma.

III. Analysis

With these preliminaries out of the way, it is possible to begin the analysis leading up to the main factorization theorems. From this point on, all matrices are assumed to be rational unless stated explicitly otherwise.

Lemma 1: A matrix $G(p)$ is analytic in the entire p plane together with its inverse (either right, left or both) if and only if it is an elementary polynomial matrix.

Proof: The "if" part is obvious. According to e) of the Smith-McMillan lemma, the analyticity of $G(p)$ for all p implies that $\psi_1(p)$ is a constant. Thus, by b) all ψ's are constant. Now note that the existence of a left or right inverse for A implies that either $n = r$ or $m = r$, respectively, and a little thought should convince the reader that the canonic form for $G^{-1}(p)$ is

$$\mathrm{diag}\left[\frac{\psi_r(p)}{e_r(p)}, \frac{\psi_{r-1}(p)}{e_{r-1}(p)}, \cdots, \frac{\psi_1(p)}{e_1(p)}\right].$$

The analyticity of $G^{-1}(p)$ in the entire p plane implies that $e_r(p) =$ constant. Invoking b) again, all e's are constant and $G(p)$ is the product of three elementary polynomial matrices, of rank r, Q.E.D.

Lemma 2: A paraconjugate unitary matrix is bounded at infinity and analytic on the entire closed $p = j\omega$ axis.

Proof: Suppose $G(p)$ is $m \times n$ and $G_*(p)G(p) = 1_n$. Thus $G^*(j\omega)G(j\omega) = 1_n$ and, writing out the diagonal elements in expanded form,

$$\sum_{r=1}^{m} |\ g_{rk}(j\omega)\ |^2 = 1, \quad (k = 1, 2, \cdots, n).$$

$$\therefore\ |\ g_{rk}(j\omega)\ | \leq 1, \quad (r = 1, 2, \cdots, m;\ k = 1, 2, \cdots, n),$$

for all ω, Q.E.D.

Lemma 3: The only regular paraconjugate unitary matrices $G(p)$ with analytic inverses in $Re\ p > 0$ are constant unitary matrices. If $G(p)$ is para-unitary it is real-orthogonal.

Proof: Suppose $G_*(p)G(p) = 1_n$, say, where $G(p)$ is a regular $m \times n$ paraconjugate unitary matrix. The analyticity of its left inverse in $Re\ p > 0$ implies that of $\bar{G}(-\bar{p})$ in the same region and therefore that of $\bar{G}(\bar{p})$ in $Re\ p < 0$. Now the poles of $\bar{G}(\bar{p})$ are the complex conjugates of those of $G(p)$. Hence $G(p)$ is analytic in the entire p plane and bounded at infinity (Lemma 2). By Liouville's theorem it must be a constant unitary matrix. If $G(p)$ is real it is real-orthogonal, Q.E.D.

Def. 5:[1] Let $G(p)$ be an $m \times n$ rational matrix of normal rank r. A decomposition of the form

$$G(p) = A(p)\ \Delta(p)B(p) \tag{7}$$

is said to be a left-standard factorization if

a_1) $\Delta(p)$ is $r \times r$, canonic and analytic together with its inverse in the entire p plane with the possible exception of a finite number of points on the $p = j\omega$ axis;

a_2) $A(p)$ is $m \times r$ and analytic together with its left inverse in $Re\ p \leq 0$;

a_3) $B(p)$ is $r \times n$ and analytic together with its right inverse in $Re\ p \geq 0$.

Interchanging A and B gives rise to a right-standard factorization. Obviously any left-standard factorization of $G(p)$ generates a right-standard factorization of $G'(p)$, $G^{-1}(p)$ and $G(-p)$. For example, $G'(p) = B'(p)\ \Delta(p)A'(p)$, etc.

It follows from the Smith-McMillan lemma that any rational matrix $G(p)$ possesses a left- or right-standard factorization. For let $G(p) = C(p)\ D(p)F(p)$ where C and F are elementary and D canonic. By factoring the e's and ψ's appearing in the diagonal elements of $D(p)$ into the product of three polynomials, the first without zeros in $Re\ p \leq 0$, the second without zeros in $Re\ p \neq 0$, and the third without zeros in $Re\ p \geq 0$, it is possible to write $D(p) = D^-(p)\ \Delta(p)\ D^+(p)$: $D^-(p)$ and its inverse are analytic in $Re\ p \leq 0$, $\Delta(p)$ and $\Delta^{-1}(p)$ in $Re\ p \neq 0$ and $D^+(p)$ and its inverse in $Re\ p \geq 0$. Now, choosing $A(p) = C(p)\ D^-(p)$ and $B(p) = D^+(p)F(p)$, it is immediate that the desired breakdown is given by $G = A \cdot \Delta B$, Q.E.D.

Suppose that $G(p)$ admits two left-standard factorizations

$$G = A\ \Delta B = A_1\ \Delta_1 B_1. \tag{8}$$

Then

$$\Delta_1^{-1} A_1^{-1} A\ \Delta = B_1 B^{-1}. \tag{9}$$

By definition the right-hand side of (9) is analytic in $Re\ p \geq 0$ and the left-hand side in $Re\ p < 0$. Thus $B_1 B^{-1}$ is analytic in the entire p plane. According to (8) the inverse of $B_1 B^{-1}$ is $\Delta^{-1} A^{-1} A_1\ \Delta_1 = BB_1^{-1}$ and is therefore also analytic in the entire p plane. By Lemma 1, $B_1 B^{-1}$ is an elementary $r \times r$ polynomial matrix $N(p)$. Similarly, $A_1^{-1} A$ is an $r \times r$ elementary polynomial matrix $M(p)$. From (8),

$$M(p)\ \Delta(p) N^{-1}(p) = \Delta_1(p).$$

Since $\Delta(p)$ and $\Delta_1(p)$ are both canonic, $\Delta(p) = \Delta_1(p)$ by the S.M. lemma. Thus,

$$M(p) = \Delta(p) N(p)\ \Delta^{-1}(p), \tag{10}$$

$$B_1(p) = N(p)B(p), \tag{11}$$

$$A_1(p) = A(p)\ \Delta(p) N^{-1}(p)\ \Delta^{-1}(p) = A(p) M^{-1}(p) \tag{12}$$

These results are summarized in Theorem 1.

Theorem 1: Let $G(p)$ possess the two left-standard factorizations $G = A\ \Delta B = A_1\ \Delta_1 B_1$. Then,

a) $\Delta(p) = \Delta_1(p)$;

b) $A_1(p) = A(p) M^{-1}(p)$ and $B_1(p) = N(p)B(p)$, where $M(p)$ and $N^{-1}(p)$ are any two $r \times r$ elementary polynomial matrices which transform $\Delta(p)$ into itself, *viz*, $M(p)\ \Delta(p) N^{-1}(p) = \Delta(p)$.

Corollary: The canonic matrix $\Delta(p)$ appearing in either a left-standard or right-standard factorization of an $m \times n$ matrix $G(p)$ of normal rank $r(G)$ is equal to the $r \times r$ identity matrix 1_r if and only if $G(p)$ is analytic and $r(G)$ is constant on the entire finite $p = j\omega$ axis. In this case, if AB and $A_1 B_1$ are any two standard factorizations of G, $A_1(p) = A(p) N^{-1}(p)$ and $B_1(p) = N(p)B(p)$, $N(p)$ being an arbitrary $r \times r$ elementary polynomial matrix.

Proof: The "if" part is immediate. Now the analyticity of $G(p)$ on the $p = j\omega$ axis implies that all the denominator polynomials in $\Delta(p)$ are unity. This, in turn, leads to the conclusion that $r(G)$ is constant on $p = j\omega$ only if all numerator polynomials in $\Delta(p)$ are unity. Thus $\Delta(p) = 1_r$. The remaining statements are a consequence of Theorem 1, part *b*), Q.E.D.

For paraconjugate hermetian matrices (see Def. 2), M and N are tied together in a very specific manner. Thus, suppose $G(p) = G_*(p)$, and let $G = A\ \Delta B$ be a left-standard factorization. Then $G(p) = G_*(p) = B_*(p)\ \Delta_*(p) A_*(p)$. Except, perhaps, for the signs of some of its diagonal elements, $\Delta_*(p)$ is also canonic, whence, from Theorem 1,

$$\Delta_*(p) = \Sigma\ \Delta(p), \tag{13}$$

where

$$\Sigma = \text{diag}\,[\epsilon_1, \epsilon_2, \cdots, \epsilon_r],\ \epsilon_k = \pm 1, \quad (k = 1, 2, \cdots, r).$$

In other words,

$$\begin{aligned} G(p) &= B_*(p)\Sigma\ \Delta(p) A_*(p) \\ &= B_*(p)\ \Delta_*(p) A_*(p) \end{aligned} \tag{13a}$$

is also a left-standard factorization. Invoking Theorem 1 again,

$$A_*(p) = N(p)B(p), \tag{14}$$

$$B_*(p) = AM^{-1}\Sigma. \tag{15}$$

$$\therefore\ A_*(p) = N(p)\Sigma M_*^{-1}(p) A_*(p). \tag{16}$$

[1] The reader is warned that this definition is not the same as that given in Goldberg and Krein [11].

Since $A_*(p)$ has a right inverse,

$$N(p) = M_*(p)\Sigma \tag{17}$$

in which $M(p)$ is any $r \times r$ elementary polynomial matrix satisfying [see (10)]

$$\Delta(p)M_*(p) = M(p)\,\Delta_*(p). \tag{18}$$

According to (13), each diagonal element of $\Delta(p)$ is either a paraconjugate or skew-paraconjugate rational function; *i.e.*, either $\Delta_{kk}(p) = \bar{\Delta}_{kk}(-\bar{p})$, or $\Delta_{kk}(p) = -\bar{\Delta}_{kk}(-\bar{p})$, $(k = 1, 2, \cdots, r)$. From (10), $|N(p)| = |M(p)|$. Thus, by (17), $|N(p)| = |\Sigma| \cdot |\overline{N(p)}| = \pm|\overline{N(p)}|$ depending on whether $|\Sigma| = \pm 1$ and $|N(p)|$ is either purely real or purely imaginary. When $G(p)$ is para-hermetian, $|N(p)|$ is real, $|\Sigma| = +1$ and the number of odd rational functions appearing in $\Delta(p)$ is even. The above statements can be made much more precise for the class of non-negative paraconjugate hermetian matrices.

Lemma 4: Let $G(p)$ be an $n \times n$ paraconjugate hermetian matrix of normal rank r which is non-negative on the real-frequency axis; *i.e.*, $\mathbf{b}^*G(j\omega)\mathbf{b} \geq 0$ for every n-vector $\mathbf{b}$ and every real ω. Then 1) its S.M. canonic form satisfies $D_*(p) = \Sigma\, D(p)$, and 2) the real-frequency zeros and poles of the diagonal elements of $D(p)$ are of even multiplicity.

Proof: Let $G(p) = C(p)\,D(p)F(p)$ be the S.M. form of $G(p)$. Since $G(p) = G_*(p)$, $C(p)\,D(p)F(p) = F_*(p)\,D_*(p)C_*(p)$. Hence, by a previous argument, $D_*(p) = \Sigma\,D(p)$ where Σ is an $r \times r$ diagonal matrix whose diagonal elements are either ± 1, and therefore each diagonal element of $D(p)$ is either paraconjugate or skew-paraconjugate. Thus any zero or pole p_0 is accompanied by a zero or pole $-\bar{p}_0$, and therefore

$$D(p) = \Sigma_1\lambda_*(p)\,\Delta(p)\lambda(p) \tag{19}$$

and

$$\Delta_*(p) = \Sigma_2\,\Delta(p), \tag{19a}$$

where $\lambda(p)$ is rational, diagonal and analytic together with its inverse in $Re\ p \geq 0$; $\Delta(p)$ is canonic, the zeros and poles of its diagonal elements being entirely confined to the $p = j\omega$ axis.

Since all the principal minors of $G(j\omega)$ are non-negative, any real-frequency pole of $G(p)$ of order k must be a pole of order k of at least one diagonal element $g_{mm}(p)$. Under the assumption that the numerators and denominators of all entries in $G(p)$ are relatively prime, $g_{mm}(j\omega) \geq 0$ implies that any one of its poles on $p = j\omega$ is of even multiplicity; *i.e.*, k is always an even integer and the denominator of $\Delta_{11}(p)$ is the square of a monic polynomial which is either paraconjugate or skew-paraconjugate.

Denote the real-frequency poles of $G(p)$ by $p = 0$, $j\omega_1, j\omega_2, \cdots, j\omega_s$, and let $l_0, l_1, l_2, \cdots, l_s$, be their highest respective multiplicities in any nondiagonal element. Define the polynomial $\mu(p)$ by

$$\mu(p) = \prod_{a=1}^{s} p^{l_0}(p - j\omega_a)^{l_a}.$$

Clearly, the only elements of $\hat{G}(p) = \mu G$ possessing real-frequency poles are diagonal. Set $D(p) = \text{diag}\,[e_1/\psi_1, e_2/\psi_2, \cdots, e_r/\psi_r]$. The S.M. canonic form for μG is

$$\hat{D}(p) = \text{diag}\left[\frac{e_1}{\hat{\psi}_1}, \frac{\hat{e}_2}{\hat{\psi}_r}, \cdots, \frac{\hat{e}_r}{\hat{\psi}_r}\right], \tag{20}$$

where $\hat{\psi}_1 = \psi_1/\mu$, and $\hat{e}_i/\hat{\psi}_i$ is $\mu e_i/\psi_i$ in lowest normalized terms, $(i = 2, 3, \cdots, r)$; $\hat{\psi}_i$ differs from ψ_i, $(i = 2, 3, \cdots, r)$, if and only if μ and ψ_i have a factor in common. Now let $\gamma_1^{(i)} \geq \gamma_2^{(i)} \geq \cdots \geq \gamma_r^{(i)}$ be the orders of $j\omega_i$, $(i = 0, 1, \cdots, s;\ \omega_0 \equiv 0)$, as a zero of $\hat{\psi}_1, \hat{\psi}_2, \cdots, \hat{\psi}_r$, respectively. Similarly, let $\sigma_1^{(i)} \geq \sigma_2^{(i)} \geq, \cdots, \geq \sigma_n^{(i)}$ be the orders of $j\omega_i$ as a pole of the diagonal elements of μG arranged in nonincreasing sequence. By Theorem 5.29 of McMillan [5],

$$\gamma_k^{(i)} = \sigma_k^{(i)}, \quad (k = 1, 2, \cdots, r;\ i = 0, 1, \cdots, s). \tag{21}$$

Thus the order of $j\omega_i$ as a zero of $\psi_k(p)$, $(i = 0, 1, \cdots, s;\ k = 1, 2, \cdots, r)$, is equal to its order as a pole of some diagonal element of $G(p)$, and is therefore an even integer. To sum up, every denominator appearing in $\Delta(p)$ is the square of a monic polynomial which is either paraconjugate or skew-paraconjugate.

As regards the numerators of $\Delta(p)$ note that from (13a) and (14),

$$[B_*^{-1}(p)G(p)B^{-1}(p)]^{-1} = N^{-1}(p)\,\Delta^{-1}(p)\Sigma_2, \tag{23}$$

so that $\Delta^{-1}(p)$ is real-frequency canonic for the paraconjugate hermetian matrix appearing on the left-hand side of (23). This matrix is also non-negative on $p = j\omega$. Consequently, all denominators of $\Delta^{-1}(p)$, and therefore all numerators of $\Delta(p)$, are the squares of either paraconjugate or skew-paraconjugate functions. Gathering everything together, $\Delta_*(p) = \Delta(p) = \theta^2(p)$, $\theta_*(p) = \Sigma_3\theta(p)$, $D_*(p) = D(p)$, $\Sigma_2 = \Sigma = 1_r$, and

$$D(p) = \Sigma_4\lambda_*(p)\theta_*(p)\theta(p)\lambda(p); \tag{24}$$

$\Sigma_4 = \Sigma_1\Sigma_3$, $\lambda(p)$ is diagonal and analytic with its inverse in $Re\ p \geq 0$, $\theta(p)$ is diagonal and analytic with its inverse in $Re\ p \neq 0$ and Σ_1, Σ_3 and Σ_4 are $r \times r$ diagonal matrices whose diagonal elements are either ± 1, Q.E.D.

Enough material is now on hand for the main theorem.

Theorem 2: Let $G(p) = G_*(p)$ be a rational $n \times n$ paraconjugate hermetian matrix of normal rank r which is non-negative on the real-frequency axis $p = j\omega$. Then, there exists an $r \times n$ rational matrix $H(p)$ such that

a_1) $G(p) = H_*(p)H(p)$.

a_2) $H(p)$ and $H^{-1}(p)$, its right inverse, are both analytic in $Re\ p > 0$.

a_3) $H(p)$ is unique up to within a constant, unitary $r \times r$ matrix multiplier on the left; *i.e.*, if $H_1(p)$ also satisfies a_1 and a_2, $H_1(p) = TH(p)$ where T is $r \times r$, constant and satisfies $T^*T = 1_r$.

a_4) Any factorization of the form $G(p) = L_*(p)L(p)$ in which $L(p)$ is $r \times n$, rational and analytic in $Re\ p > 0$, is given by $L(p) = V(p)H(p)$, $V(p)$ being an arbitrary, rational, regular $r \times r$ paraconjugate unitary matrix.

a_5) If $G(p)$ is analytic on the finite $p = j\omega$ axis, $H(p)$ is analytic in a right semi-infinite strip $Re\ p > -\tau$, $\tau > 0$.

a_6) If $G(p)$ is analytic and $r(G)$ is invariant on the finite $p = j\omega$ axis, $H^{-1}(p)$ is analytic in a right semi-infinite strip $Re\ p > -\tau_1$, $\tau_1 > 0$.

a_7) If $G(p)$ is real, $H(p)$ and $V(p)$ are real and T is real-orthogonal.

Proof: Consider statement a_3) first, and let $H(p)$ and $H_1(p)$ be two matrices satisfying a_1) and a_2). Then

$$H_*(p)H(p) = H_{1*}(p)H_1(p) \tag{25}$$

$$\therefore\ V_*(p)V(p) = 1_r \tag{26}$$

where $V(p) = H_1(p)H^{-1}(p)$ is obviously analytic in $Re\ p > 0$; *i.e.*, $V(p)$ is a regular $r \times r$ paraconjugate unitary matrix. But from (25),

$$V(p) = H_{1*}^{-1}(p)H_*(p), \tag{27}$$

and is therefore also analytic in $Re\ p \leq 0$. By Lemma 3, $V(p)$ is a constant $r \times r$ unitary matrix T. Hence $H_1(p) = TH(p)$, Q.E.D.

The proof of a_4) proceeds along the same lines and is omitted.

To prove the existence of an $H(p)$ with the properties a_1) and a_2) is of course the difficult part.

Step 1: Reduce $G(p)$ to the S.M. canonic form. One procedure for doing this is the following: Assuming that all entries in G are relatively prime, write

$$G(p) = g^{-1}(p)\tilde{G}(p), \tag{28}$$

where $g(p)$ is the normalized lowest common multiple of all denominators appearing in $G(p)$ and $\tilde{G}(p)$ is a polynomial matrix. It is easily shown [5] that $g(p) = \psi_1(p)$. $\tilde{G}(p)$ is now reduced to its Smith form by the technique described in Gantmacher [8]; *i.e.*,

$$\tilde{G}(p) = \tilde{C}(p)\tilde{E}(p)\tilde{F}(p), \tag{29}$$

where $\tilde{C}(p)$ and $\tilde{F}(p)$ are $n \times n$ elementary polynomial matrices and

$$\tilde{E}(p) = \text{diag}\,[\tilde{e}_1(p), \tilde{e}_2(p), \cdots, \tilde{e}_r(p), 0, 0, \cdots, 0]. \tag{30}$$

The $\tilde{e}$'s are monic polynomials arranged so that $\tilde{e}_i$ divides $\tilde{e}_{i+1}$, $(i = 1, 2, \cdots, r-1)$. Let

$$J = \begin{bmatrix} 1_r \\ \hline 0 \end{bmatrix}\begin{matrix} r \\ n-r \end{matrix}, \quad (\text{width } r). \tag{31}$$

Then $C(p) = \tilde{C}(p)J$ and $F(p) = J'\tilde{F}(p)$ are $n \times r$ and $r \times n$ elementary polynomial matrices, respectively. Moreover,

$$\tilde{G}(p) = C(p)E(p)F(p), \tag{32}$$

where

$$E(p) = \text{diag}\,[\tilde{e}_1, \tilde{e}_2, \cdots, \tilde{e}_r]. \tag{33}$$

If now $D(p)$ is defined by

$$D(p) = \text{diag}\left[\frac{\tilde{e}_1}{g}, \frac{\tilde{e}_2}{g}, \cdots, \frac{\tilde{e}_r}{g}\right], \tag{34}$$

each element being normalized and in lowest terms, $\tilde{e}_1 = e_1$, $\psi_1 = g$ and the S.M. form for $G(p)$ is $G = CDF$.

Step 2: According to Lemma 4,

$$D(p) = \Sigma\lambda_*(p)\ \Delta(p)\lambda(p), \tag{35}$$

where

1) $\lambda(p)$ is $r \times r$, diagonal and analytic, together with $\lambda^{-1}(p)$ in $Re\ p \geq 0$;
2) $\Delta_*(p) = \Delta(p) = \theta^2(p)$ in which all diagonal elements of $\theta(p)$ are either paraconjugate or skew-paraconjugate. Furthermore, $\Delta(p)$ is canonic and analytic in $Re\ p \neq 0$;
3) Σ is an $r \times r$ diagonal matrix with diagonal elements ± 1.

Let

$$A(p) = C(p)\Sigma\lambda_*(p), \tag{36}$$

$$B(p) = \lambda(p)F(p). \tag{37}$$

Then

$$G(p) = A(p)\ \Delta(p)B(p) \tag{38}$$

is a left-standard factorization of $G(p)$.

Step 3: By (13a) and (14) of the corollary to Lemma 3,

$$B_*^{-1}(p)G(p)B^{-1}(p) = \theta_*^2(p)N(p) = \Delta(p)N(p), \tag{39}$$

where $N(p) = (n_{rk})$ is an $r \times r$ elementary polynomial matrix such that [see (17) and (18)]

$$\Delta(p)N(p)\ \Delta^{-1}(p) = M(p) \tag{40}$$

is also elementary. From (39),

$$I_*(p)G(p)I(p) = \theta_*(p)N(p)\theta^{-1}(p), \tag{41}$$

$$I(p) = B^{-1}(p)\theta^{-1}(p). \tag{42}$$

Hence

$$\tilde{M}(p) \equiv \theta_*(p)N(p)\theta^{-1}(p) \tag{43}$$

is $r \times r$, paraconjugate hermetian and non-negative on the $p = j\omega$ axis. Actually a good deal more is true. Observe that (40) and the canonic nature of $\Delta(p)$ imply that $n_{rk}(p)$ is divisible by the polynomial $\Delta_{kk}(p)/\Delta_{rr}(p)$, $k \geq r$. Since

$\Delta_{kk}(p) = \theta_k^2(p)$, $(k = 1, 2, \cdots, r)$, $n_{rk}(p)$ must be divisible by the polynomial

$$f_{rk}^2(p) = \frac{\theta_k^2(p)}{\theta_r^2(p)}, \qquad k \geq r$$

and, *a fortiori*, by $f_{rk}(p) = \theta_k(p)/\theta_r(p) = \pm\theta_{k*}(p)/\theta_r(p)$, $k \geq r$. This suffices to establish that $\tilde{M}(p)$ is polynomial. But $|\tilde{M}(p)| = \pm |N(p)| =$ constant; *i.e.*, $\tilde{M}(p)$ is a positive paraconjugate hermetian $r \times r$ elementary polynomial matrix. The next step is to demonstrate that

$$\tilde{M}(p) = P_*(p)P(p), \tag{44}$$

$P(p)$ being an $r \times r$ elementary polynomial matrix. After this is achieved, the desired factorization for $G(p)$ is obtained as $G = H_*(p)H(p)$ with

$$\begin{aligned} H(p) &= P(p)\theta(p)B(p) \\ &= P(p)\theta(p)\lambda(p)F(p) \qquad (45) \\ &= P(p)\, D^+(p)F(p) \end{aligned}$$

where

$$D^+(p) = \theta(p)\lambda(p). \tag{46}$$

By straightforward algebra,

$$\begin{aligned} H_*(p)H(p) &= F_*(p)\lambda_*(p)\theta_*(p)P_*(p)P(p)\theta(p)\lambda(p)F(p) \\ &= F_*(p)\lambda_*(p)\theta_*^2(p)N(p)\lambda(p)F(p) \\ &= B_*(p)\,\Delta(p)N(p)B(p) \\ &= G(p). \end{aligned}$$

The ingenious algorithm to be described in Step 4 for factoring a positive, elementary polynomial paraconjugate hermetian matrix is due to Oono and Yasuura and first appeared in a now classic paper [6] dealing with the synthesis of passive n ports. Another such application may be found in Youla [7].

Step 4: Because of the positive nature of $\tilde{M}(j\omega)$, all its diagonal elements are paraconjugate and positive on $p = j\omega$. Let $2\delta_1 \leq 2\delta_2 \leq, \cdots, \leq 2\delta_r$ be the degrees of these diagonal entries arranged in nondecreasing order. The δ's are non-negative integers. Again invoking the positive character of $\tilde{M}(j\omega)$, it follows that no element in $\tilde{M}(p)$ has degree exceeding $2\delta_r$. Thus $\delta_r = 0$ if and only if $\tilde{M}(p)$ is a constant hermetian positive definite $r \times r$ matrix, in which case it can be written as P^*P by any number of standard techniques. The Gauss algorithm is as good as any [8]. Excluding this relatively trivial situation, $\delta_r > 0$.

By interchanging rows and columns it may be assumed that the diagonal elements $(\tilde{M})_{11}$, $(\tilde{M})_{22}$, $\cdots$, $(\tilde{M})_{rr}$ possess the degrees $2\delta_{11}$, $2\delta_{22}$, $\cdots$, $2\delta_{rr}$, respectively. Call the rearranged matrix $\tilde{M}_1(p)$. Then there exists a permutation matrix Q such that

$$\tilde{M}_1(p) = Q'\tilde{M}(p)Q. \tag{47}$$

$\tilde{M}_1(p)$ is also elementary, paraconjugate hermetian and positive.

Define a nonincreasing sequence of non-negative integers $\sigma_1, \sigma_2, \cdots, \sigma_r$ by

$$\sigma_i = \delta_r - \delta_i, \qquad (i = 1, 2, \cdots, r), \tag{48}$$

and the $r \times r$ diagonal matrix $\Omega(p)$ by

$$\Omega(p) = \operatorname{diag}\,[p^{\sigma_1}, p^{\sigma_2}, \cdots, p^{\sigma_r}]. \tag{49}$$

Note that $\sigma_r = 0$. The $r \times r$ matrix

$$\tilde{M}_2(p) = \Omega_*(p)\tilde{M}_1(p)\Omega(p) \tag{50}$$

is polynomial, paraconjugate hermetian and positive. Moreover, all its diagonal elements have the same degree $2\delta_r$. It is clear that

$$|\tilde{M}_2(p)| = O(p^{2\sigma}), \tag{51}$$

$$\sigma = \sigma_1 + \sigma_2 + \cdots + \sigma_{r-1}. \tag{52}$$

From (48)

$$\sigma \leq (r-1)\,\delta_r. \tag{53}$$

$\tilde{M}_2(p)$ may be expanded as a polynomial in p with constant matrix coefficients:

$$\tilde{M}_2(p) = T_0 + pT_1 + \cdots + p^{2\delta_r}T_{2\delta_r}. \tag{54}$$

Since $\tilde{M}_{2*}(p) = \tilde{M}_2(p)$, $T_{2\delta_r} = T^*_{2\delta_r}$, $T_{2\delta_r-1} = -T^*_{2\delta_r-1}$, $\cdots$, $T_1 = -T_1^*$ and $T_0 = T_0^*$. The T's are constant hermetian or skew-hermetian $r \times r$ matrices.

The important observation is that $T_{2\delta_r}$ is singular; *i.e.*, $|T_{2\delta_r}| = 0$, for otherwise (54) would yield

$$|\tilde{M}_2(p)| = O(p^{2r\delta_r}),$$

which contradicts (51) and (53). This deduction implies that $T_{2\delta_r}$ contains a principal minor Γ of order $\nu \times \nu$, $1 \leq \nu < r$, located in its upper left-hand corner (Fig. 1)

$$T_{2\delta_r} = \begin{array}{c} \\ \nu \\ 1 \\ \\ \end{array} \begin{array}{c} \begin{array}{ccc} \nu & 1 & r-(\nu+1) \end{array} \\ \left[\begin{array}{c|c|c} \Gamma & \mathbf{x} & \\ \hline \mathbf{x}^* & t_{\nu+1,\,\nu+1} & T_a \\ \hline \multicolumn{2}{c|}{T_a^*} & T_b \end{array}\right] \end{array} \begin{array}{l} \\ \nu+1 \\ \\ r-(\nu+1) \end{array}$$

$\mathbf{x}$ = a ν-dimensional column vector.

Γ = nonsingular hermetian $\nu \times \nu$ matrix.

$$\tilde{\Gamma} = \left[\begin{array}{c|c} \Gamma & \mathbf{x} \\ \hline \mathbf{x}^* & t_{\nu+1,\,\nu+1} \end{array}\right] = (\nu+1) \times (\nu+1) \text{ singular hermetian matrix.}$$

Fig. 1—Structure of $T_{2\delta_r}$.

which is nonsingular and such that the minor $\tilde{\Gamma}$ created by adding the $(\nu+1)$th row and column to Γ is singular: Suppose this assertion is false. Then since the (1, 1) element in $T_{2\delta_r}$ is not zero (remember that all diagonal entries in $\tilde{M}_2(p)$ have degree $2\delta_r$), the upper left-hand corner 2×2, 3×3, $\cdots$, $r \times r$ minors of $T_{2\delta_r}$ must all be nonsingular. But the last minor is precisely $|T_{2\delta_r}| = 0$, a contradiction, Q.E.D.

By adding a proper linear combination of the first ν rows of $T_{2\delta_r}$ to the $(\nu+1)$th row and the conjugate linear combination of the first ν columns to the $(\nu+1)$th, $t_{\nu+1,\nu+1}$ is reduced to zero, and no other diagonal element is affected. Hence, for the correct choice of constant $r \times r$ nonsingular matrix Q_1,

$$\tilde{T}_{2\delta_r} = Q_1^* T_{2\delta_r} Q_1 \tag{55}$$

has a zero element in the $(\nu+1, \nu+1)$ place. From (54),

$$\tilde{M}_3(p) \equiv Q_1^* \tilde{M}_2(p) Q_1 = \sum_{i=0}^{2\delta_r} (Q_1^* T_i Q_1) p^i \tag{56}$$

has a diagonal element in the $(\nu+1, \nu+1)$ position of degree less than $2\delta_r$.

The matrix

$$\tilde{M}_4(p) = \Omega_*^{-1}(p) \tilde{M}_3(p) \Omega^{-1}(p) \tag{57}$$

is paraconjugate hermetian, positive and elementary. Only the latter statement needs proof. According to (50), $(\tilde{M}_2)_{kl}$ is divisible by $p^{\sigma_k+\sigma_l}$, and according to (56) and the definition of Q_1, $\tilde{M}_3(p)$ differs from $\tilde{M}_2(p)$ only in its $(\nu+1)$th row and column. More specifically,

$$(\tilde{M}_3)_{k,\nu+1} = (\tilde{M}_2)_{k,\nu+1} + \sum_{i=1}^{\nu} \alpha_i (\tilde{M}_2)_{ki}, \tag{58}$$

$(k = 1, 2, \cdots, r)$, the α's being scalars. By construction $\sigma_1 \geq \sigma_2 \geq \cdots, \geq \sigma_r$. Thus every term on the right-hand side of (58) is divisible by $p^{\sigma_k+\sigma_{\nu+1}}$, $(k = 1, 2, \cdots, r)$. The same considerations apply to the $(\nu+1)$th row, whence, for all k and l, $(\tilde{M}_3)_{kl}$ is divisible by $p^{\sigma_k+\sigma_l}$, and $\tilde{M}_4(p)$ is a polynomial matrix. Since

$$|\tilde{M}_4(p)| = \pm |Q_1 \bar{Q}_1 Q^2| \cdot |\tilde{M}(p)| = \text{constant},$$

$\tilde{M}_4(p)$ is elementary, Q.E.D.

But $\tilde{M}_4(p)$ is simpler than $\tilde{M}_1(p)$ because the degree of its $(\nu+1, \nu+1)$ entry is at least two less than the one in the same place in the latter, while all other corresponding diagonal elements have the same degree. Consequently, after one cycle of the algorithm,

$$\tilde{M}(p) = R_{1*}(p) \tilde{M}_4(p) R_1(p), \tag{59}$$

where

$$R_1(p) = \Omega(p) Q_1^{-1} \Omega^{-1}(p) Q^{-1} \tag{60}$$

is an elementary polynomial matrix and $\tilde{M}_4(p)$ is at least two degrees less than $\tilde{M}(p)$. That $R_1(p)$ is elementary is almost obvious by inspection. The reader is invited to supply a formal proof for himself. After a maximum of $\delta = r\delta_r$ cycles, $\tilde{M}(p)$ is reduced to a constant hermetian positive definite matrix $\tilde{M}_{4\delta} = C^*C$, so that finally,

$$\tilde{M}(p) = P_*(p) P(p),$$

where

$$P(p) = C R_\delta(p) R_{\delta-1}(p) \cdots R_1(p). \tag{61}$$

This completes the proof of parts a_1) and a_2).

As regards a_5), note that the analyticity of $G(p)$ on $p = j\omega$ implies that $\theta(p)$ is polynomial, which in turn implies that $D^+(p) = \theta(p)\lambda(p)$ is analytic in a strip $Re\ p > -\tau$, $\tau > 0$. This strip is completely determined by $\lambda(p)$. Thus $H(p) = P(p)\ D^+(p) F(p)$ is also analytic in $Re\ p > -\tau$.

Under the hypotheses of a_6), $\theta(p) = 1_r$ (see the corollary to Theorem 1), and

$$H^{-1}(p) = F^{-1}(p) \lambda^{-1}(p) P^{-1}(p) \tag{62}$$

is analytic in some strip $Re\ p > -\tau_1$, $\tau_1 > 0$. By d) of the S.M. lemma, the reality of $G(p)$ permits all associated matrices to be chosen real and therefore $H(p)$, $V(p)$ and T are real by construction. This terminates the proof of Theorem 2.

Corollary 1: Any factorization of the form $G(p) = L_*(p) L(p)$ in which $L(p)$ is $m \times n$, $m \geq r(G)$, is given by

$$L(p) = V(p) \left[\frac{1_r}{0_{m-r,r}} \right] H(p) \tag{62a}$$

where $V(p)$ is an arbitrary $m \times m$ paraconjugate unitary matrix.

Proof: Clearly, $L(p)$ must be of the form $L(p) = U(p) H(p)$, $U(p)$ being an $m \times r$ rational paraconjugate unitary matrix. The result now follows by choosing $V(p)$ to be any $m \times m$ paraconjugate unitary matrix with $U(p)$ incorporated into its first r columns; *i.e.*,

$$U(p) = V(p) \left[\frac{1_r}{0_{m-r,r}} \right], \tag{62b}$$

$V(p)$ an arbitrary $m \times m$ paraconjugate unitary matrix, Q.E.D.

Corollary 2: If $G(p)$ is polynomial $H(p)$ is polynomial.

Proof: If $G(p)$ is polynomial $D^+(p)$ is polynomial. Thus, by (45), $H(p)$ is polynomial, Q.E.D.

Example 1: To see how the above theorem works, consider the nontrivial 3×3 para-hermetian matrix

$$G(p) = \left[\begin{array}{c|c|c} \dfrac{1}{1-p^2} & \dfrac{1}{p(1-p^2)} & 0 \\ \hline -\dfrac{1}{p(1-p^2)} & \dfrac{p^2-2}{p^2(1-p^2)} & \dfrac{1}{2p(1-p^2)} \\ \hline 0 & \dfrac{-1}{2p(1-p^2)} & \dfrac{1}{1-p^2} \end{array} \right]. \tag{63}$$

It is easily verified that all principal minors are positive on the real-frequency axis. Hence $G(j\omega) > 0$.

Step 1: The normalized lowest common multiple of all denominators is

$$g(p) = \psi_1(p) = p^4 - p^2 = p^2(p^2 - 1) \tag{64}$$

d

$$gG = \hat{G}(p) = \begin{bmatrix} -p^2 & -p & 0 \\ p & 2-p^2 & -p/2 \\ 0 & p/2 & -p^2 \end{bmatrix}. \tag{65}$$

rst, the procedure described in Gantmacher [8][2] is ed to reduce $\hat{G}(p)$ to Smith canonic form:

a) Interchange the first and second columns. This amounts to multiplying $\hat{G}$ on the right by

$$O_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{66}$$

d the result is

$$\hat{G}_1 = \begin{bmatrix} -p & -p^2 & 0 \\ 2-p^2 & p & -p/2 \\ p/2 & 0 & -p^2 \end{bmatrix}. \tag{67}$$

b) Multiply the first row of $\hat{G}_1$ by $-p$ and add to the second. This is accomplished by multiplying $\hat{G}_1$ on the left with

$$S_1 = \begin{bmatrix} 1 & 0 & 0 \\ -p & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{68}$$

d the result is

$$\hat{G}_2 = \begin{bmatrix} -p & -p^2 & 0 \\ 2 & p+p^3 & -p/2 \\ p/2 & 0 & -p^2 \end{bmatrix}. \tag{69}$$

c) Interchange the first and second rows of $\hat{G}_2$ and multiply the first row by $\frac{1}{2}$. This is accomplished by multiplying $\hat{G}_2$ on the left with

$$S_2 = \begin{bmatrix} 0 & 1/2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{70}$$

d the result is

$$\hat{G}_3 = \begin{bmatrix} 1 & \dfrac{p+p^3}{2} & -p/4 \\ -p & -p^2 & 0 \\ p/2 & 0 & -p^2 \end{bmatrix}. \tag{71}$$

d) Now multiply the first row by p and $-p/2$ in turn and add to the second and third, respectively. This is achieved by multiplying $\hat{G}_3$ on the left with

$$S_3 = \begin{bmatrix} 1 & 0 & 0 \\ p & 1 & 0 \\ -p/2 & 0 & 1 \end{bmatrix}, \tag{72}$$

the result being

$$\hat{G}_4 = \begin{bmatrix} 1 & \dfrac{p+p^3}{2} & -p/4 \\ 0 & \dfrac{p^4-p^2}{2} & -p^2/4 \\ 0 & \dfrac{p^2+p^4}{4} & -\dfrac{7}{8}p^2 \end{bmatrix}. \tag{73}$$

e) Multiply the first column by $-(p+p^3)/2$ and $p/4$ and add, in the same order, to the second and third. This is accomplished by multiplying $\hat{G}_4$ on the right with

$$O_2 = \begin{bmatrix} 1 & -\dfrac{p+p^3}{2} & p/4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{74}$$

and the result is

$$\hat{G}_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{p^4-p^2}{2} & -p^2/4 \\ 0 & -\dfrac{p^2+p^4}{4} & -\dfrac{7}{8}p^2 \end{bmatrix}. \tag{75}$$

f) Interchange the second and third columns; multiply the second row by $-7/2$ and add to the third; multiply the second column by $2p^2$ and add to the third; multiply the second column by -2 and add to the third, and finally multiply the second column by -4 and the third by $-\frac{1}{2}$.

The end product is

$$\hat{G}_6 = S_4\hat{G}_5O_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & p^2 & 0 \\ 0 & 0 & p^4 - \frac{3}{4}p^2 \end{bmatrix}, \tag{76}$$

where

$$S_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -7/2 & 1 \end{bmatrix}, \tag{77}$$

and

$$O_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1/2 \\ 0 & -4 & 1-p^2 \end{bmatrix}. \tag{78}$$

[2] See pp. 134–139.

Letting

$$C^{-1}(p) = S_4S_3S_2S_1 = \begin{bmatrix} -p/2 & 1/2 & 0 \\ 1 - p^2/2 & p/2 & 0 \\ -\frac{7}{2} + 2p^2 & -2p & 1 \end{bmatrix} \tag{79}$$

and

$$F^{-1} = O_1O_2O_3 = \begin{bmatrix} 0 & 0 & -1/2 \\ 1 & -p & p/2 \\ 0 & -4 & 1 - p^2 \end{bmatrix}, \tag{80}$$

$G = CDF$ where $D = g^{-1}\hat{G}_6$ is the S.M. canonic form:

$$D(p) = \operatorname{diag}\left[\frac{1}{p^2(p^2 - 1)}, \frac{1}{p^2 - 1}, \frac{p^2 - \frac{3}{4}}{p^2 - 1}\right]. \tag{81}$$

Step 2: Clearly

$$\lambda(p) = \operatorname{diag}\left[\frac{1}{p+1}, \frac{1}{p+1}, \frac{p + \frac{\sqrt{3}}{2}}{p+1}\right], \tag{82}$$

$$\theta(p) = \operatorname{diag}\left[\frac{1}{p}, 1, 1\right], \tag{83}$$

$$D^+(p) = \lambda(p)\theta(p)$$

$$= \operatorname{diag}\left[\frac{1}{p(p+1)}, \frac{1}{p+1}, \frac{p + \frac{\sqrt{3}}{2}}{p+1}\right], \tag{84}$$

$$A(p) = C(p) \operatorname{diag}\left[\frac{1}{p-1}, \frac{1}{p-1}, \frac{p - \frac{\sqrt{3}}{2}}{p+1}\right], \tag{85}$$

$$B(p) = \lambda(p)F(p), \tag{86}$$

and [see (14)],

$$N(p) = A'(-p)B^{-1}(p). \tag{87}$$

Step 3: By direct matrix multiplication,

$$\tilde{M}(p) = \theta(-p)N(p)\theta^{-1}(p)$$

$$= \begin{bmatrix} 2 - p^2 & p^2 & 0 \\ p^2 & 14 - p^2 & 4p - 2\sqrt{3} \\ 0 & -4p - 2\sqrt{3} & 1 - p^2 \end{bmatrix}. \tag{88}$$

It is easily verified that $\tilde{M}(p)$ is elementary, para-hermetian and positive.

Step 4: The remaining task is to factor $\tilde{M}(p)$ into $P'(-p)P(p)$, $P(p)$ being polynomial. Observe that all diagonal elements are of second degree, *i.e.*, $2\delta_2 = 2$. Thus $\tilde{M}(p) = \tilde{M}_2(p)$ and [see (54)],

$$T_2 = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{89}$$

Since the upper left-hand corner 2×2 minor is singular,

$$\Gamma = [-1], \tag{90}$$

$$\tilde{\Gamma} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \tag{91}$$

and $t_{2,2} = -1$ (refer to Fig. 1 for the meaning of the symbols).

The result of adding the first row and column in $\tilde{M}_2$ to the second row and column, respectively, is

$$Q_1'\tilde{M}_2(p)Q_1 = \tilde{M}_3(p) = \tilde{M}_4(p)$$

$$= \begin{bmatrix} 2 - p^2 & 2 & 0 \\ 2 & 16 & 4p - 2\sqrt{3} \\ 0 & -4p - 2\sqrt{3} & 1 - p^2 \end{bmatrix}, \tag{92}$$

where

$$Q_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{93}$$

The (2, 2) diagonal element has been reduced in degree and the first cycle of the algorithm is over.

The second cycle is begun by arranging the diagonal elements of $\tilde{M}_4(p)$ in nondecreasing order of degree from upper left-hand corner to lower right:

$$Q_2'\tilde{M}_4(p)Q_2 = \tilde{M}_5(p)$$

$$= \begin{bmatrix} 16 & 2 & 4p - 2\sqrt{3} \\ 2 & 2 - p^2 & 0 \\ -4p - 2\sqrt{3} & 0 & 1 - p^2 \end{bmatrix}, \tag{94}$$

$$Q_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{95}$$

The diagonal elements of $\tilde{M}_5(p)$ must now be made equidegree:

$$\Omega_2(-p)\tilde{M}_5(p)\Omega_2(p) = \tilde{M}_6(p)$$

$$= \begin{bmatrix} -16p^2 & -2p & -4p^2 + 2\sqrt{3}\,p \\ 2p & 2 - p^2 & 0 \\ -4p^2 - 2\sqrt{3}\,p & 0 & 1 - p^2 \end{bmatrix}, \tag{96}$$

$$\Omega_2(p) = \operatorname{diag}\,[p, 1, 1]. \tag{97}$$

The coefficient matrix of p^2 is

$$T_{2,1} = \begin{bmatrix} -16 & 0 & -4 \\ 0 & -1 & 0 \\ -4 & 0 & -1 \end{bmatrix}, \tag{98}$$

ence

$$\Gamma_1 = \begin{bmatrix} -16 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \tilde{\Gamma}_1 = T_{2,1}. \tag{99}$$

ultiply the first row and column of $\tilde{M}_6(p)$ by $-\frac{1}{4}$ and d to the third row and column, respectively. Thus

$$\tilde{M}_6(p)Q_3 = \tilde{M}_7(p)$$

$$= \begin{bmatrix} -16p^2 & -2p & 2\sqrt{3}\,p \\ 2p & 2-p^2 & -p/2 \\ -2\sqrt{3}\,p & p/2 & 1 \end{bmatrix}, \tag{100}$$

$$Q_3 = \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{101}$$

e second cycle is brought to an end by performing e operation inverse to (96):

$$^{-1}(-p)\tilde{M}_7(p)\Omega_2^{-1}(p) = \tilde{M}_8(p)$$

$$= \begin{bmatrix} 16 & 2 & -2\sqrt{3} \\ 2 & 2-p^2 & -p/2 \\ -2\sqrt{3} & p/2 & 1 \end{bmatrix}. \tag{102}$$

Only one more cycle is necessary. Interchange the t two rows and columns of $\tilde{M}_8(p)$:

$$\tilde{M}_8Q_4 = \tilde{M}_9(p) = \begin{bmatrix} 16 & -2\sqrt{3} & 2 \\ -2\sqrt{3} & 1 & p/2 \\ 2 & -p/2 & 2-p^2 \end{bmatrix}, \tag{103}$$

$$Q_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}. \tag{103a}$$

w make all diagonal elements of $\tilde{M}_9(p)$ equidegree:

$$(-p)\tilde{M}_9(p)\Omega_3(p) = \tilde{M}_{10}(p)$$

$$= \begin{bmatrix} -16p^2 & 2p^2\sqrt{3} & -2p \\ 2p^2\sqrt{3} & -p^2 & -p^2/2 \\ 2p & -p^2/2 & 2-p^2 \end{bmatrix}, \tag{104}$$

$$\Omega_3(p) = \operatorname{diag}[p, p, 1]. \tag{105}$$

us,

$$T_{2,2} = \begin{bmatrix} -16 & 2\sqrt{3} & 0 \\ 2\sqrt{3} & -1 & -1/2 \\ 0 & -1/2 & -1 \end{bmatrix}, \tag{106}$$

$$\Gamma_2 = \begin{bmatrix} -16 & 2\sqrt{3} \\ 2\sqrt{3} & -1 \end{bmatrix}, \qquad \tilde{\Gamma}_2 = T_{2,2}. \tag{107}$$

Multiply the first column of $\tilde{M}_{10}(p)$ by $-\sqrt{3}/4$, the second column by -2, sum them and add the result to the third. Do the same for the rows. Then,

$$Q_5'\tilde{M}_{10}(p)Q_5 = \tilde{M}_{11}(p) = \begin{bmatrix} -16p^2 & 2p^2\sqrt{3} & -2p \\ 2p^2\sqrt{3} & -p^2 & 0 \\ 2p & 0 & 2 \end{bmatrix}, \tag{108}$$

$$Q_5 = \begin{bmatrix} 1 & 0 & -\frac{\sqrt{3}}{4} \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}. \tag{109}$$

Lastly,

$$\Omega_3^{-1}(-p)\tilde{M}_{11}(p)\Omega_3^{-1}(p) = \tilde{M}_{12}(p)$$

$$= \begin{bmatrix} 16 & -2\sqrt{3} & 2 \\ -2\sqrt{3} & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix}, \tag{110}$$

a constant, real, symmetric positive-definite matrix.

Using formula (42) of Gantmacher [8],[3] it is easy to decompose $\tilde{M}_{12}(p)$ into a product of triangular factors: $\tilde{M}_{12}(p) = C'C$, where

$$C = \begin{bmatrix} 4 & \frac{\sqrt{3}}{2} & 1/2 \\ 0 & 1/2 & \frac{\sqrt{3}}{2} \\ 0 & 0 & 1 \end{bmatrix}. \tag{111}$$

Collecting all matrices and carrying out the calculation gives $\tilde{M}(p) = P'(-p)P(p)$, where

$$P(p) = C\Omega_3(p)Q_5^{-1}\Omega_3^{-1}(p)Q_4^{-1}\Omega_2(p)Q_3^{-1}\Omega_2^{-1}(p)Q_2^{-1}Q_1^{-1} \tag{112}$$

$$= \begin{bmatrix} 1/2 & 7/2 & p - \frac{\sqrt{3}}{2} \\ p + \frac{\sqrt{3}}{2} & -p - \frac{\sqrt{3}}{2} & 1/2 \\ 1 & -1 & 0 \end{bmatrix}. \tag{113}$$

Finally, the desired expression for $H(p)$ is

$$P(p)\,D^+(p)F(p) = H(p) =$$

$$= \begin{bmatrix} 0 & \frac{1}{2p(p+1)} & -\frac{1}{p+1} \\ 0 & \frac{p+\sqrt{3}/2}{p(p+1)} & 0 \\ \frac{1}{p+1} & \frac{1}{p(p+1)} & 0 \end{bmatrix}, \tag{114}$$

[3] See vol. 1, p. 38.

and is evidently analytic together with its inverse in $Re\ p > 0$. Of course, many of the calculations appearing in Example 1 can be abbreviated and have been carried out in their entirety in order to give the reader a clear picture of the mechanism underlying the algorithm.

The distinguishing feature of Theorem 2 is that it yields a factor matrix $H(p)$ that is analytic *together with its right-inverse* in $Re\ p > 0$. In problems of the Wiener-Hopf type this property of $H(p)$ is of crucial importance. On the other hand, some network problems, such as the synthesis of lumped, passive n ports [9] merely require an $H(p)$ analytic in $Re\ p > 0$, with no restrictions on the analytic character of $H^{-1}(p)$. In this case, it is possible to exhibit a decomposition $G(p) = H_*(p)H(p)$ in which $H(p)$ is upper-triangular and to give explicit formulas for the computation of its elements.

Theorem 3: Let $G(p)$ be a rational $n \times n$ paraconjugate hermetian matrix of normal rank n which is non-negative on the real-frequency axis $p = j\omega$. Then there exists a rational upper-triangular $n \times n$ matrix $H(p)$, such that

a_1) $G(p) = H_*(p)H(p)$.

a_2) $H(p)$ is rational and analytic in $Re\ p > 0$.

a_3) Under the assumption that all entries in $H(p)$ are relatively prime, the elements of any row have no common zeros in $Re\ p > 0$.

a_4) $H(p)$ is uniquely determined up to within a constant diagonal unitary matrix multiplier on the left; *i.e.*, if $H_1(p)$ is upper-triangular and also satisfies a_1), a_2) and a_3), $H_1(p) = TH(p)$, where $T = \operatorname{diag}[e^{i\phi_1}, e^{i\phi_2}, \cdots, e^{i\phi_n}]$, the ϕ's being real constants.

a_5) Any factorization of the form $G(p) = L_*(p)L(p)$ in which $L(p)$ is upper-triangular and analytic in $Re\ p > 0$ is given by $L(p) = V(p)H(p)$, where $V(p)$ is a regular, diagonal rational $n \times n$ paraconjugate unitary matrix.

a_6) If $G(p)$ is real, $H(p)$ can be chosen real and T real-orthogonal. Furthermore, $L(p)$ real implies $V(p)$ real.

Proof: Consider a_4) first, and suppose that $H(p)$ and $H_1(p)$ are two upper-triangular matrices possessing properties $a_1)-a_3$). Then, by a_1),

$$H_*(p)H(p) = H_{1*}(p)H_1(p). \tag{115}$$

$$\therefore\ H_{1*}^{-1}(p)H_*(p) = H_1(p)H^{-1}(p) \equiv V(p) \tag{116}$$

and

$$V_*(p)V(p) = 1_n. \tag{116a}$$

Now (116) shows that $V(p)$ must be both lower- and upper-triangular and hence diagonal. Thus $H_1(p) = V(p)H(p)$ where $V(p)$ is a diagonal paraconjugate unitary matrix.

By hypothesis, $H_1(p)$ is regular so that any right-hand pole of a diagonal element in $V(p)$ must be a common zero of all entries in the corresponding row in $H(p)$. But according to a_3) this situation is impossible whence it follows that all the diagonal elements in $V(p)$ are regular paraconjugate unitary functions, *i.e.*, regular "Blaschke" products. Any such product $b(p)$ has the representation

$$b(p) = e^{i\phi} \prod_{r=1}^{l} \frac{p - p_r}{p + \bar{p}_r}, \qquad Re\ p_r > 0, \qquad (r = 1, 2, \cdots, l); \tag{117}$$

the zeros of $b(p)$ are all restricted to the right-hand p plane.

On the other hand, if any $b(p)$ appearing in $V(p)$ has a zero in $Re\ p > 0$, this zero is common to all elements in the corresponding row of $H_1(p)$, since the analyticity of $H(p)$ in $Re\ p > 0$ excludes any possibility of cancellation. This contradicts a_3), and therefore $V(p)$ is constant and of the form

$$V(p) = \operatorname{diag}[e^{i\phi_1}, e^{i\phi_2}, \cdots, e^{i\phi_n}],$$

Q.E.D.

Assertion a_6) is obvious and the proof of a_5) is almost identical with that for a_4) and is omitted.

It now remains to demonstrate the existence of an upper-triangular factorization $H(p)$ with the attributes a_1), a_2) and a_3). Actually it is only necessary to construct an upper-triangular matrix $H(p)$ analytic in $Re\ p > 0$ satisfying a_1. For suppose such an $H(p)$ is available. Define $b_r(p)$, $(r = 1, 2, \cdots, n)$, to be that regular Blaschke product formed with all the common right-hand zeros (multiplicities included) of the rth row of $H(p)$. Set

$$V(p) = \operatorname{diag}[b_1(p), b_2(p), \cdots, b_n(p)].$$

Then $\hat{H}(p) = V_*(p)H(p)$ is upper-triangular and meets conditions $a_1)-a_3$).

The concise notation

$$A\begin{pmatrix} i_1 i_2 & \cdots & i_m \\ k_1 k_2 & \cdots & k_m \end{pmatrix}$$

is used to denote the minor of the matrix A formed with the rows numbered i_1, i_2, $\cdots$ i_m and the columns $k_1, k_2, \cdots, k_m$. Let

$$G(p) = g^{-1}(p)\hat{G}(p),$$

where $g(p)$ is the normalized lowest common multiple of all denominators appearing in $G(p)$. Then $g(p) = \psi_1(p)$ (see the factorization theorem) and

$$g(p) = \epsilon t_*(p)t(p), \qquad \epsilon = \pm 1, \tag{118}$$

the polynomial $t(p)$ being devoid of zeros in $Re\ p > 0$. Hence,

$$\tilde{G}(p) \equiv \epsilon \hat{G}(p) = t_*(p)G(p)t(p) \tag{119}$$

is an $n \times n$ polynomial, non-negative, paraconjugate hermetian matrix of normal rank n.

According to Theorem 1 of Gantmacher [8],[4] $\tilde{G}(p)$ can be represented as a product of a lower-triangular

[4] See p. 35.

trix $S(p)$ and an upper-triangular matrix $\tilde{H}(p)$; *i.e.*, $p) = S(p)\tilde{H}(p)$, where $S = (s_{rk})$, $\tilde{H} = (\tilde{h}_{rk})$ and

$$s_{rr}(p)\tilde{h}_{rr}(p) = \frac{\tilde{G}_r(p)}{\tilde{G}_{r-1}(p)}, \qquad (r = 1, 2, \cdots, n), \tag{120}$$

$$s_{rk}(p) = s_{kk}(p)\frac{\tilde{G}\begin{pmatrix}1\ 2\ \cdots\ k-1,\ r\\ 1\ 2\ \cdots\ k-1,\ k\end{pmatrix}}{\tilde{G}\begin{pmatrix}1\ 2\ \cdots\ k\\ 1\ 2\ \cdots\ k\end{pmatrix}}, \tag{121}$$

d

$$(p) = \tilde{h}_{kk}(p)\frac{\tilde{G}\begin{pmatrix}1\ 2\ \cdots\ k-1,\ k\\ 1\ 2\ \cdots\ k-1,\ r\end{pmatrix}}{\tilde{G}\begin{pmatrix}1\ 2\ \cdots\ k\\ 1\ 2\ \cdots\ k\end{pmatrix}},$$

$$(r = k, k+1, \cdots, n;\ k = 1, 2, \cdots, n), \tag{122}$$

which ($\tilde{G}_0 \equiv 1$)

$$\tilde{G}_r(p) \equiv \tilde{G}\begin{pmatrix}1\ 2\ \cdots\ r\\ 1\ 2\ \cdots\ r\end{pmatrix} \not\equiv 0, \tag{123}$$

r $(r = 1, 2, \cdots, n)$. These latter inequalities are a nsequence of the positive character of $\tilde{G}(j\omega)$ and the sumption that its normal rank is n.

Now all the $\tilde{G}$'s and $\tilde{G}_r$'s are polynomials in p. By pothesis, $\tilde{G}_*(p) = \tilde{G}(p)$ which in turn implies that

$$\begin{pmatrix}1\ 2\ \cdots\ k-1,\ r\\ 1\ 2\ \cdots\ k-1,\ k\end{pmatrix} = \tilde{G}_*\begin{pmatrix}1\ 2\ \cdots\ k-1,\ k\\ 1\ 2\ \cdots\ k-1,\ r\end{pmatrix},$$

$$(r, k = 1, 2, \cdots, n). \tag{124}$$

particular,

$$\tilde{G}_r(p) = \tilde{G}_{r*}(p), \qquad (r = 1, 2, \cdots, n). \tag{125}$$

nce $\tilde{G}(j\omega) \geq 0$, $\tilde{G}_r(j\omega) \geq 0$, $(r = 1, 2, \cdots, n)$. Thus ery paraconjugate polynomial $\tilde{G}_r(p)$ can be factored:

$$\tilde{G}_r(p) = y_{r*}(p)y_r(p), \qquad (r = 1, 2, \cdots, n), \tag{126}$$

e polynomials $y_r(p)$ being free of zeros in $Re\ p > 0$. Set ($y_0 \equiv 1$)

$$\tilde{h}_{rr}(p) = \frac{y_{r*}(p)}{y_{r-1}(p)}, \tag{127}$$

d

$$s_{rr}(p) = \frac{y_r(p)}{y_{r-1*}(p)}, \qquad (r = 1, 2, \cdots, n). \tag{128}$$

is obvious by (126) that (120) is satisfied, that $\tilde{h}_{rr}(p)$ analytic in $Re\ p > 0$, and that $s_{rr}(p) = \tilde{h}_{rr*}(p)$, $= 1, 2, \cdots, n)$. From (122) and (121),

$$\tilde{h}_{kr}(p) = \frac{\tilde{G}\begin{pmatrix}1\ 2\ \cdots\ k-1,\ k\\ 1\ 2\ \cdots\ k-1,\ r\end{pmatrix}}{y_{k-1}(p)y_k(p)} \tag{129}$$

and

$$s_{rk}(p) = \frac{\tilde{G}\begin{pmatrix}1\ 2\ \cdots\ k-1,\ r\\ 1\ 2\ \cdots\ k-1,\ k\end{pmatrix}}{y_{k-1*}(p)y_{k*}(p)} = \tilde{h}_{kr*}(p),$$

$$(r = k, k+1, \cdots, n;\ k = 1, 2, \cdots, n). \tag{130}$$

Hence $\tilde{H}(p) = (\tilde{h}_{rk})$ is upper-triangular, analytic in $Re\ p > 0$ and obeys $\tilde{G}(p) = \tilde{H}_*(p)\tilde{H}(p)$. The matrix

$$H(p) = t^{-1}(p)\tilde{H}(p) \tag{131}$$

meets the desired requirements a_1) and a_2), Q.E.D.

Corollary: Let $G(p)$ be an $n \times n$ rational paraconjugate hermetian matrix of normal rank r which is non-negative on the $p = j\omega$ axis. Then there exist rational matrices $H_1(p)$, $A(p)$, an $n \times n$ permutation matrix Q and a regular Blaschke product $b(p)$, such that

b_1) $H_1(p)$ is $r \times r$, nonsingular, upper triangular and analytic in $Re\ p > 0$. Moreover, the elements in any one of its rows have no common zeros in $Re\ p > 0$;

b_2) $A(p)$ is $r \times (n - r)$ and $b(p)A(p)$ is analytic in $Re\ p > 0$;

b_3) The $r \times n$ matrix

$$H(p) = b(p)H_1(p)[1_r \mid A(p)] \tag{131a}$$

is analytic in $Re\ p > 0$ and satisfies $H_*(p)H(p) = Q'G(p)Q$;

b_4) For the same choice of Q, $H_1(p)$ is uniquely determined up to within a constant diagonal $r \times r$ unitary matrix left-multiplier $V(p)$;

b_5) $G(p)$ real implies that $H_1(p)$ and $A(p)$ are real and V is real-orthogonal;

b_6) If $r = n$, Q may be chosen equal to 1_n and $b(p) = 1$.

Proof: Since $G(p)$ is paraconjugate hermetian and of normal rank r, it possesses at least one nonsingular principal minor of order r. By permuting rows and columns, this minor can be shifted to the upper left-hand corner. Thus, for the proper choice of permutation matrix Q,

$$Q'G(p)Q = \begin{array}{cc} \left[\begin{array}{c|c} G_1(p) & G_2(p) \\ \hline G_{2*}(p) & G_3(p) \end{array}\right] & \begin{array}{l} r \\ n-r \end{array} \\ \begin{array}{cc} r & n-r \end{array} & \end{array}, \tag{131b}$$

where $G_1(p)$ is of normal rank r. In addition, $G_{3*}(p) = G_3(p)$ and $G_{1*}(p) = G_1(p)$ are both non-negative on $p = j\omega$. By the definition of rank,

$$G_2(p) = G_1(p)A(p)$$

and

$$G_3(p) = G_{2*}(p)A(p) = A_*(p)G_1(p)A(p),$$

$A(p)$ being a rational $r \times n - r$ matrix. Hence,

$$Q'G(p)Q = M_*(p)G_1(p)M(p), \tag{131c}$$

where

$$M(p) = [1_r \mid A(p)], \qquad A(p) = G_1^{-1}(p)G_2(p). \tag{131d}$$

Let $g(p)$ be the lowest common multiple of all denominators appearing in $A(p)$. Then

$$Q'G(p)Q = \frac{1}{g_*(p)g(p)} \tilde{M}_*(p)G_1(p)\tilde{M}(p), \tag{131e}$$

the matrix $\tilde{M}(p) = g(p)M(p)$ being polynomial. By Theorem 3, there exists a matrix $H_1(p)$ with the property b_1) satisfying $H_{1*}(p)H_1(p) = G_1(p)$. Again, $g_*(p)g(p)$ is paraconjugate hermetian and non-negative on $p = j\omega$ and so admits the Hurwitz factorization

$$g_*(p)g(p) = h_*(p)h(p),$$

the polynomial $h(p)$ being free of zeros in $Re\ p > 0$. Evidently $g(p) = b(p)h(p)$ where $b(p)$ is a regular all-pass factor. Consequently,

$$Q'G(p)Q = H_*(p)H(p), \text{ where}$$

$$H(p) = \frac{g(p)}{h(p)} H_1(p)[1_r \mid A(p)]$$

$$= b(p)H_1(p)[1_r \mid A(p)], \tag{131f}$$

and $b(p)A(p)$ are analytic in $Re\ p > 0$ by actual construction.

Now suppose that

$$Q'G(p)Q = H_*(p)H(p) = \hat{H}_*(p)\hat{H}(p)$$

in which

$$H(p) = b(p)H_1(p)[\overset{r}{1_r} \mid \overset{n-r}{A(p)}]$$

and

$$\hat{H}(p) = \hat{b}(p)\hat{H}_1(p)[\overset{r}{1_r} \mid \overset{n-r}{\hat{A}(p)}].$$

Then,

$$[1_r \mid A(p)]_* H_{1*}(p)H_1(p)[1_r \mid A(p)]$$

$$= [1_r \mid \hat{A}(p)]_* \hat{H}_{1*}(p)\hat{H}_1(p)[1_r \mid \hat{A}(p)].$$

$$\therefore H_{1*}H_1 = \hat{H}_{1*}\hat{H}_1$$

$$\therefore \hat{H}_{1*}^{-1}H_{1*} = \hat{H}_1 H_1^{-1}$$

is both upper- and lower-triangular, and, using what should by now be a familiar argument, it is concluded that $\hat{H}_1(p) = V(p)H_1(p)$, where $V(p)$ is an $r \times r$, constant, diagonal unitary matrix; b_5) is immediate and as for b_6) note that $r = n$ implies $G_1(p) = G(p)$ so that $Q = 1_n$ and $b(p) = 1$ are admissible, Q.E.D.

Example 2: As an illustration, consider once more the 3×3 para-hermetian matrix $G(p)$ in (63). From (64), $g(p) = p^4 - p^2 = t(-p)t(p)$ where $t(p) = p(1 + p)$, whence $\epsilon = +1$ and

$$\hat{G}(p) = \tilde{G}(p) = \begin{bmatrix} -p^2 & -p & 0 \\ p & 2 - p^2 & -p/2 \\ 0 & p/2 & -p^2 \end{bmatrix}. \tag{65a}$$

By direct computation,

$$\tilde{G}_1(p) = -p^2 = y_1(-p)y_1(p); \qquad y_1(p) = p; \tag{132}$$

$$\tilde{G}_2(p) = p^4 - p^2 = y_2(-p)y_2(p);$$

$$y_2(p) = p(1 + p); \tag{133}$$

$$\tilde{G}_3(p) = -p^6 + \tfrac{3}{4}p^4 = y_3(-p)y_3(p);$$

$$y_3(p) = p^2\left(\frac{\sqrt{3}}{2} + p\right); \tag{134}$$

$$\tilde{G}\begin{pmatrix}1\\2\end{pmatrix} = -p; \tag{135}$$

$$\tilde{G}\begin{pmatrix}1\\3\end{pmatrix} = 0; \tag{136}$$

$$\tilde{G}\begin{pmatrix}1\ 2\\1\ 3\end{pmatrix} = p^3/2. \tag{137}$$

Using the formulas (127)–(137),

$$\tilde{h}_{11}(p) = \frac{y_1(-p)}{1} = -p; \tag{138}$$

$$\tilde{h}_{22}(p) = \frac{y_2(-p)}{y_1(p)} = p - 1; \tag{139}$$

$$\tilde{h}_{33}(p) = \frac{y_3(-p)}{y_2(p)} = \frac{p\left(\frac{\sqrt{3}}{2} - p\right)}{1 + p}; \tag{140}$$

$$\tilde{h}_{12}(p) = \frac{\tilde{G}\begin{pmatrix}1\\2\end{pmatrix}}{1 \cdot y_1(p)} = -1; \tag{141}$$

$$\tilde{h}_{13}(p) = \frac{\tilde{G}\begin{pmatrix}1\\3\end{pmatrix}}{1 \cdot y_1(p)} = 0; \tag{142}$$

$$\tilde{h}_{23}(p) = \frac{\tilde{G}\begin{pmatrix}1\ 2\\1\ 3\end{pmatrix}}{y_1(p)y_2(p)} = \frac{p}{2(1 + p)}. \tag{143}$$

Thus $G(p) = H_T'(-p)H_T(p)$, where

$$H_T(p) = t^{-1}(p)\tilde{H}(p)$$

$$= \left[\begin{array}{c|c|c} -\dfrac{1}{p+1} & -\dfrac{1}{p(p+1)} & 0 \\ \hline 0 & \dfrac{p-1}{p(p+1)} & \dfrac{1}{2(p+1)^2} \\ \hline 0 & 0 & \dfrac{\frac{\sqrt{3}}{2} - p}{(1+p)^2} \end{array}\right]. \tag{144}$$

is obvious that $H_T^{-1}(p)$ is not analytic in $Re\ p > 0$.

The reader will perhaps find it interesting to compare $_T(p)$ with the $H(p)$ of (114) and to assure himself that e matrix

$$(p) = H_T(p)H^{-1}(p)$$

$$= \begin{bmatrix} 0 & 0 & -1 \\ -\dfrac{1}{2(p+1)} & \dfrac{p - \dfrac{\sqrt{3}}{2}}{p+1} & 0 \\ \dfrac{p - \dfrac{\sqrt{3}}{2}}{1+p} & \dfrac{\dfrac{\sqrt{3}}{2} - p}{2(1+p)\left(p + \dfrac{\sqrt{3}}{2}\right)} & 0 \end{bmatrix} \qquad (145)$$

regular and para-unitary, in agreement with a_4) of heorem 2.

The matrix

$$\tilde{H}_T(p) = \begin{bmatrix} -\dfrac{1}{p+1} & -\dfrac{1}{p(p+1)} & 0 \\ 0 & \dfrac{p-1}{p(p+1)} & \dfrac{1}{2(p+1)^2} \\ 0 & 0 & \dfrac{p + \dfrac{\sqrt{3}}{2}}{(1+p)^2} \end{bmatrix} \qquad (146)$$

also a regular upper-triangular solution of $G(p) =$ $'(-p)H(p)$, but unlike (144) the elements in any row e relatively prime with respect to right-hand zeros. learly,

$$H_T(p) = V(p)\tilde{H}_T(p)$$

here

$$V(p) = \operatorname{diag}\left[1, 1, \frac{\dfrac{\sqrt{3}}{2} - p}{\dfrac{\sqrt{3}}{2} + p}\right].$$

IV. Applications

Problem 1: Solve the integral equation (1) by the iener-Hopf technique subject to the following re- rictions:

w_1) $G(p)$ is rational and has the properties listed under (3);

w_2) $G(p)$ and $G^{-1}(p)$ are both analytic in a strip $-\eta < Re\ p < \eta$, $\eta > 0$;

w_3)

$$\mathbf{E}(p) \equiv \int_{-\infty}^{\infty} \mathbf{e}(t)e^{-pt}\,dt \qquad (147)$$

has a strip of convergence intersecting the interval $-\eta < Re\ p < \eta$.

Solution: Let

$$\mathbf{y}(t) \equiv \int_0^{\infty} K(t-\tau)\mathbf{W}(\tau)\,d\tau - \mathbf{e}(t),$$

$$(-\infty < t < \infty). \qquad (148)$$

Then $\mathbf{y}(t) = \mathbf{0}$, $t > 0$ and its bilateral Laplace transform

$$\mathbf{Y}(p) = \int_{-\infty}^{+\infty} \mathbf{y}(t)e^{-pt}\,dt \qquad (149)$$

is analytic in some left half plane.

Transformation of both sides of (148) yields

$$\mathbf{Y}(p) = G(p)\mathbf{F}(p) - \mathbf{E}(p) \qquad (150)$$

in some common strip; $\mathbf{F}(p)$ is the transform of $\mathbf{W}(\tau)$; *i.e.*,

$$\mathbf{F}(p) = \int_{-\infty}^{+\infty} \mathbf{W}(\tau)e^{-p\tau}\,d\tau. \qquad (151)$$

The physical realizability of the filters

$$W_1(\tau), W_2(\tau), \cdots, W_n(\tau)$$

implies that $\mathbf{F}(p)$ is also analytic in some right half plane.

According to Theorem 2, $G(p) = H_*(p)H(p)$ where $H(p)$ is real, rational and analytic, together with its inverse in $-\eta < Re\ p$. From (150),

$$H_*^{-1}(p)\mathbf{Y}(p) = H(p)\mathbf{F}(p) - H_*^{-1}(p)\mathbf{E}(p). \qquad (152)$$

In general, $H_*^{-1}(p)\mathbf{E}(p)$ is not analytic in either half plane, and one must resort to the usual artifice of decomposing it into the sum

$$H_*^{-1}(p)\mathbf{E}(p) = \{H_*^{-1}(p)\mathbf{E}(p)\}_+ + \{H_*^{-1}(p)\mathbf{E}(p)\}_- \qquad (153)$$

in which the first factor on the right is analytic in a half plane $Re\ p > -\mu$, $\mu > 0$, and the second in $Re\ p < \mu$. Inserting (153) into (152) and rearranging gives

$$H_*^{-1}(p)\mathbf{Y}(p) + \{H_*^{-1}(p)\mathbf{E}(p)\}_-$$

$$= H(p)\mathbf{F}(p) - \{H_*^{-1}(p)\mathbf{E}(p)\}_+. \qquad (154)$$

The right-hand side of (154) is analytic in some strip $Re\ p > -\mu_1$, $\mu_1 > 0$, and the left-hand side in $Re\ p < +\mu_1$. Thus the right-hand side is an entire matrix function of p. The simplest solution is obtained by setting this entire function equal to the zero matrix. Thus

$$\mathbf{F}(p) = H^{-1}(p)\{H_*^{-1}(p)\mathbf{E}(p)\}_+, \qquad (155)$$

and its strip of convergence is some right half plane. The only aim of the above derivation is to indicate how the factorization idea enters into the Wiener-Hopf technique; most of the details concerning rigor have been purposely omitted. Suffice it to say that these details are not too difficult to fill in for G's meeting conditions w_1)$-w_3$). Formula (155) highlights in a most emphatic manner the importance of the requirement that $H^{-1}(p)$ as well as $H(p)$ be analytic in $Re\ p > -\eta$. The filters defined by (155) are not necessarily stable.

The case in which $G(p)$ is of normal rank less than n is singular and not important as far as the physical applications are concerned because it represents a situation in which the noise can be completely obliterated by an appropriate selection and interconnection of differentiators. For, if $r(G) < n$ there exists a nontrivial polynomial n vector $\mathbf{F}(p) = (f_1, f_2, \cdots, f_n)'$, such that $G(p)\mathbf{F}(p) = \mathbf{0}_n$ and the weighting functions

$$W_k(\tau) = f_k\left(\frac{d}{dt}\right), \qquad (k = 1, 2, \cdots, n), \tag{156}$$

do the trick.

Another interesting question is that of generalizing the concept of "flat" noise to the multivariable case. Fortunately, this turns out to be unexpectedly simple: The k-dimensional noise process $\mathbf{n}(t) = (n_1, n_2, \cdots, n_k)'$ is said to be flat or "white" if its associated spectral density matrix $G(p)$ is an elementary polynomial matrix. Thus its entries are polynomial in p and its determinant is a nonzero constant independent of p. One justification for this definition is the following. Suppose it is desired to design a k-channel "matched" filter [2]. The appropriate integral equation is

$$\int_0^\infty K(t - \tau)\mathbf{W}(\tau)\, d\tau = \mathbf{s}(t - t_0), \qquad t > 0, \tag{157}$$

where $\mathbf{s}(t) = (s_1, s_2, \cdots, s_k)'$ is the column vector of the known channel pulse shapes, $s_1(t), s_2(t), \cdots, s_k(t)$, and t_0 is the detection instant. Transforming both sides of (157) over the doubly infinite range $(-\infty < t < \infty)$ gives

$$\begin{aligned} \mathbf{F}(p) &= e^{-pt_0}G^{-1}(p)\mathbf{S}(-p); \\ \mathbf{S}(p) &\equiv \int_{-\infty}^{+\infty} e^{-pt}\mathbf{s}(t)\, dt. \end{aligned} \tag{158}$$

As a rule, the $\mathbf{F}(p)$ described in (158) cannot be made physically realizable no matter how large a delay t_0 is incorporated into the design. There is one notable exception, however, and this occurs when $G(p)$ is an elementary polynomial matrix and $\mathbf{s}(t)$ is of finite epoch; *i.e.*, when $\mathbf{s}(t) = \mathbf{0}$ for $t < -T$, $|T| < \infty$. To see this, let $G^{-1}(p) = (l_{mn})$, the l's being polynomials in p. The operational inverse of (158) yields the weighting functions

$$W_r(\tau) = \sum_{m=1}^{k} l_{rm}\left(\frac{d}{dt}\right)s_m(\tau - t_0), \quad (r = 1, 2, \cdots, k). \tag{159}$$

If $t_0 \geq T$, $W_r(\tau) = 0$, $\tau < 0$, $(r = 1, 2, \cdots, k)$, and realizability has been achieved at the expense of system delay. Eq. (158) generalizes Dwork's well-known single-channel result [10].

Problem 2: Given an $n \times n$ rational matrix $A(p)$ of normal rank r, exhibit a factorization of the form $A(p) = V(p)H(p)$, where

1) $V(p)$ is an $n \times r$ paraconjugate unitary rational matrix, and
2) $H(p)$ is $r \times n$, rational and analytic together with its right inverse in $Re\ p > 0$.

Solution: The paraconjugate hermetian matrix $G(p) = A_*(p)A(p)$ is $n \times n$, of normal rank r and positive on $p = j\omega$. By Theorem 2, Corollary 1, there exists an $r \times n$ rational matrix $H(p)$ analytic together with its right inverse $H^{-1}(p)$ in $Re\ p > 0$, such that $G(p) = H_*(p)H(p)$ and $A(p) = V(p)H(p)$, $V(p)$ being an $n \times r$ paraconjugate unitary matrix, Q.E.D. Note that $V(p)$ is analytic in $Re\ p > 0$ if an only if $A(p)$ is analytic in $Re\ p > 0$. Moreover, $H(p)$ and $V(p)$ are unique up to within a constant $r \times r$ unitary matrix multiplier on the left and right, respectively. Lastly, $1_n - A_*(j\omega)A(j\omega) \geq 0$ implies that $1_n - H_*(j\omega)H(j\omega) \geq 0$. Thus, it is possible to factor every rational matrix into the product of a "matrix all-pass" $V(p)$ and a "minimum-phase" matrix $H(p)$ without destroying either its passive or rational character.

The next problem bears on the structure of lumped, passive, lossless n ports [7].

Problem 3: Investigate the structure of rational $n \times n$ paraconjugate unitary matrices $V(p)$.

Solution: Suppose that $V_*(p)V(p) = 1_n$, and let

$$D(p) = \operatorname{diag}\left[\frac{e_1(p)}{\psi_1(p)}, \frac{e_2(p)}{\psi_2(p)}, \cdots, \frac{e_n(p)}{\psi_n(p)}\right] \tag{160}$$

be its S.M. canonic form. Then the e's and ψ's are monic polynomials such that $e_1 \mid e_2 \mid \cdots \mid e_n(p)$ and $\psi_n \mid \psi_{n-1} \mid \cdots \mid \psi_1(p)$. The notation $f \mid g$ means that f divides g. In addition, $e_r(p)$ and $\psi_r(p)$ are relatively prime, $(r = 1, 2, \cdots, n)$. By definition, there exist two elementary $n \times n$ polynomial matrices $A(p)$ and $B(p)$, such that

$$V(p) = A(p)\, D(p)B(p). \tag{161}$$

Since $V_*(p) = V^{-1}(p)$,

$$B_*(p)\, D_*(p)A_*(p) = B^{-1}(p)\, D^{-1}(p)A^{-1}(p). \tag{162}$$

Now, except for possible plus and minus signs, $D_*(p)$ is already in canonic form, while the S.M. canonic form corresponding to $D^{-1}(p)$ is

$$\operatorname{diag}\left[\frac{\psi_n(p)}{e_n(p)}, \frac{\psi_{n-1}(p)}{e_{n-1}(p)}, \cdots, \frac{\psi_1(p)}{e_1(p)}\right],$$

and is achieved by merely permuting the rows and columns in $D^{-1}(p)$. By the uniqueness part of the S.M. lemma,

$$e_r(p) = \epsilon_r\psi_{n-r+1*}(p), \qquad (r = 1, 2, \cdots, n), \tag{163}$$

the ϵ's being either ± 1. Hence, the S.M. canonic form of $V(p)$ may be written as

$$D(p) = \Sigma \operatorname{diag}\left[\frac{\psi_{n*}(p)}{\psi_1(p)}, \frac{\psi_{n-1*}(p)}{\psi_2(p)}, \cdots, \frac{\psi_{1*}(p)}{\psi_n(p)}\right], \tag{164}$$

where

$$\Sigma = \operatorname{diag}[\epsilon_1, \epsilon_2, \cdots, \epsilon_n] \tag{164a}$$

and

$$\psi_{r*}(p) \text{ is prime to } \psi_{n-r+1}(p), \quad (r = 1, 2, \cdots, n). \tag{164b}$$

nce $V = ADB$,

$$| V(p) | = \text{constant} \times \prod_{r=1}^{n} \frac{\psi_{r*}(p)}{\psi_{n-r+1}(p)}. \tag{165}$$

Let $\psi_1(p)$ possess the distinct zeros $p_1, p_2, \cdots, p_\nu$, ith respective multiplicities $r_{11}, r_{12}, \cdots, r_{1\nu}$. Then, nce $\psi_n | \psi_{n-1} | \cdots | \psi_1$,

$$\begin{aligned} \psi_1(p) &= (p - p_1)^{r_{11}}(p - p_2)^{r_{12}} \cdots (p - p_\nu)^{r_{1\nu}}, \\ \psi_2(p) &= (p - p_1)^{r_{21}}(p - p_2)^{r_{22}} \cdots (p - p_\nu)^{r_{2\nu}}, \\ &\vdots \\ \psi_n(p) &= (p - p_1)^{r_{n1}}(p - p_2)^{r_{n2}} \cdots (p - p_\nu)^{r_{n\nu}}, \end{aligned} \tag{166}$$

here

$$\begin{aligned} r_{11} &\ge r_{21} \ge \cdots \ge r_{n1}, \\ r_{12} &\ge r_{22} \ge \cdots \ge r_{n2}, \\ &\vdots \\ r_{1\nu} &\ge r_{2\nu} \ge \cdots \ge r_{n\nu}. \end{aligned} \tag{167}$$

hose factors appearing in tableau (166) with nonzero ponents are called the elementary divisors of $V(p)$, nd the integers r_{kl}, $(k = 1, 2, \cdots, n; l = 1, 2, \cdots, \nu)$ e its indices. The total indices are the ν integers

$$r_k = \sum_{i=1}^{n} r_{ik}, \qquad (k = 1, 2, \cdots, \nu). \tag{168}$$

Now suppose that $| V(p) |$ = a constant independent p. From (165)

$$\psi_1(p)\psi_2(p) \cdots \psi_n(p) = \text{constant} \times \psi_{1*}(p)\psi_{2*}(p) \cdots \psi_{n*}(p), \tag{169}$$

, using (166) and (168),

$$\prod_{k=1}^{\nu} (p - p_k)^{r_k} = \pm \prod_{k=1}^{\nu} (p + \bar{p}_k)^{r_k}. \tag{170}$$

his implies that every zero p_k is accompanied by the ro $-\bar{p}_k$, and their associated total indices are equal. Since (p) is analytic on $p = j\omega$, $p_k \ne -\bar{p}_k$, $(k = 1, 2, \cdots, \nu)$. hus a paraconjugate unitary matrix has constant de- rminant if and only if any pole p_k has the same total dex as the pole $-\bar{p}_k$. It is an immediate corollary that ny regular $V(p)$ with constant determinant must be a nstant unitary matrix. The restriction that $\psi_{r*}(p)$ be rime to $\psi_{n-r+1}(p)$ imposes some further structure limi- tions. Thus $\psi_n(p) \equiv 1$ irrespective of the choice of (p). For example, if $n = 2$,

$$\psi_1(p) = \psi_{1*}(p), \tag{171}$$

$$\psi_2(p) = 1, \tag{172}$$

nd

$$D(p) = \operatorname{diag}\,[\psi_1^{-1}(p), \psi_1(p)]. \tag{173}$$

$n = 3$ there are several possibilities which are best xplained by dividing the indices $(r_{11}, r_{12}, \cdots, r_{1\nu})$ into two classes $(r_{11}, r_{12}, \cdots, r_{1\nu/2})$ and $(\bar{r}_{11}, \bar{r}_{12}, \cdots, \bar{r}_{1\nu/2})$, writing

$$\psi_1(p) = \prod_{i=1}^{\nu/2} (p - p_i)^{r_{1i}}(p + \bar{p}_i)^{\bar{r}_{1i}} \tag{174}$$

and considering what the situation must be like with respect to a single zero, say, p_1.

Recall that $\psi_{2*}(p)$ must be prime to $\psi_2(p)$. Thus, if $\psi_2(p)$ contains the factor $(p - p_1)$, it cannot contain the factor $(p + \bar{p}_1)$. Suppose, for definiteness, that $\bar{r}_{11} \ge r_{11}$. Then

$$\begin{aligned} \psi_1(p) &= (p - p_1)^{r_{11}}(p + \bar{p}_1)^{\bar{r}_{11}}, \\ \psi_2(p) &= (p - p_1)^{r_{21}}, \\ \psi_3(p) &= 1, \end{aligned} \tag{175}$$

where $r_{11} + r_{21} = \bar{r}_{11}$, $r_{21} \le r_{11}$. If $\bar{r}_{11} > 2r_{11}$ this latter requirement of equal total indices is impossible to meet. Therefore, any pole p_0 of multiplicity $r_0 > 0$ of a para- conjugate unitary matrix with constant determinant must be accompanied by the pole $-\bar{p}_0$ with multiplicity $\bar{r}_0$, where $0 < \bar{r}_0 \le 2r_0$.[5]

The canonic form of a paraconjugate unitary matrix is completely delineated in (164)–(164b). Conversely, given a

$$D(p) = \operatorname{diag}\left[\frac{\psi_{n*}(p)}{\psi_1(p)}, \frac{\psi_{n-1*}(p)}{\psi_2(p)}, \cdots, \frac{\psi_{1*}(p)}{\psi_n(p)}\right] \tag{176}$$

in which the ψ's are monic polynomials satisfying

a) $\psi_n | \psi_{n-1} | \psi_{n-2} | \cdots | \psi_1$, and
b) $\psi_{r*}(p)$ is prime to $\psi_{n-r+1}(p)$, $(r = 1, 2, \cdots, n)$,

does there exist a paraconjugate unitary matrix whose canonic form (up to within plus and minus signs) is $D(p)$? A complete and simple answer is available for regular matrices.

Theorem 4: The matrix $D(p)$ is the canonic form (up to within plus and minus signs) of a regular paraconjugate unitary matrix $V(p)$ if and only if $\psi_n | \psi_{n-1} | \cdots | \psi_1$ and $\psi_1(p)$ is a strict Hurwitz polynomial.

Proof: The "if" part is obvious. As regards the "only if" part consider the paraconjugate unitary matrix

$$\hat{V}(p) = \operatorname{diag}\left[\frac{\psi_{1*}(p)}{\psi_1(p)}, \frac{\psi_{2*}(p)}{\psi_2(p)}, \cdots, \frac{\psi_{n*}(p)}{\psi_n(p)}\right]. \tag{177}$$

Since $\psi_1(p)$ is strict Hurwitz and $\psi_n | \psi_{n-1} | \cdots | \psi_1$, all ψ's are strict Hurwitz and $\psi_{r*}(p)$ is automatically prime to $\psi_r(p)$, $(r = 1, 2, \cdots, n)$. Now the canonic form of $\hat{V}(p)$ is

$$\operatorname{diag}\left[\frac{\theta_{n*}(p)}{\theta_1(p)}, \frac{\theta_{n-1*}(p)}{\theta_2(p)}, \cdots, \frac{\theta_{1*}(p)}{\theta_n(p)}\right],$$

the polynomials $\theta_1, \theta_2, \cdots, \theta_n(p)$, possessing the proper- ties *a*) and *b*) listed under (176). By either direct argument or an appeal to Theorem 5.29 of McMillan [5], it is

[5] The author wishes to take this opportunity to point out that the part of the footnote appearing on p. 194 of Youla [7] which asserts that every para-unitary matrix with constant determinant is a constant matrix is incorrect.

easily established that $\psi_r(p) = \theta_r(p)$, $(r = 1, 2, \cdots, n)$. Thus (177) is a regular paraconjugate unitary matrix with the desired canonic form (176), Q.E.D.

It now follows that the most general regular paraconjugate unitary matrix $V(p)$ with the canonic form (176) is given by

$$V(p) = A(p)\hat{V}(p)B(p), \tag{178}$$

where $A(p)$ and $B(p)$ are two elementary polynomial matrices. A method for choosing $A(p)$ and $B(p)$ is the subject of Theorem 5.

Theorem 5: An elementary polynomial matrix $A(p)$ is the left-hand factor of a $V(p)$ defined by (178) if and only if the matrix $G(p) = \hat{V}_*(p)A_*(p)A(p)\hat{V}(p)$ is polynomial.

Proof: From (178)

$$G(p) = \hat{V}_*(p)A_*(p)A(p)\hat{V}(p) = B_*^{-1}(p)B^{-1}(p).$$

Hence $G(p)$ is an elementary polynomial matrix. Conversely, let $G(p)$ be polynomial. Since $| G(p) | = | A_*A | \cdot | \hat{V}_*\hat{V} | = | A_*A | =$ constant, $G(p)$ is actually elementary. Clearly, $G(p)$ is paraconjugate hermetian and positive on $p = j\omega$ and it follows, by Theorem 2, that there exists an elementary polynomial matrix $B(p)$, such that $G(p) = B_*^{-1}(p)B^{-1}(p)$; *i.e.*, the matrix $V(p) = A(p)\hat{V}(p)B(p)$ is paraconjugate unitary, Q.E.D.

Corollary: Let $D(p)$ have the properties [176, a) and b)] and let $A(p)$ be an elementary polynomial matrix, such that $D_*(p)A_*(p)A(p)D(p)$ is polynomial. Then there exists an elementary polynomial matrix $B(p)$, such that $V(p) = A(p)D(p)B(p)$ is paraconjugate unitary. The S.M. canonic form of $V(p)$ is $D(p)$.

Theorem 6 [6]: Let

$$\Psi(p) = \text{diag}\,[\psi_1(p), \psi_2(p) \cdots, \psi_n(p)],$$

where the ψ's are monic, $\psi_n \mid \psi_{n-1} \mid \cdots \mid \psi_1(p)$, and $\psi_1(p)$ is strict Hurwitz. Let $A(p)$ be an arbitrary elementary polynomial matrix. There exist two elementary polynomial matrices $P(p)$ and $F(p)$, such that

$$V(p) = A(p)\Psi_*'(p)F(p)\Psi^{-1}(p)P(p) \tag{179}$$

is a regular paraconjugate unitary matrix with the S.M. canonic form (176).

Proof: Consider the positive, polynomial, paraconjugate hermetian matrix $G(p) = \Psi(p)A_*(p)A(p)\Psi_*(p)$. The key observation is that $G(p)$ and $\Psi_*(p)\,\Psi(p) \equiv \hat{G}(p)$ possess the same S.M. canonic form. To prove this, it is necessary to show that the greatest normalized common divisors of all 1-row, 2-row, $\cdots$, n-row minors of $G(p)$ and $\hat{V}(p)$ are identical. Obviously, the greatest common divisor of all r-row minors of $\hat{G}(p)$ is

$$\hat{d}_r(p) = \theta_r(p)\theta_{r*}(p), \tag{180}$$

where

$$\theta_r(p) = \psi_{n-r+1}(p)\psi_{n-r+2}(p) \cdots \psi_n(p), \quad (r = 1, 2, \cdots, n). \tag{181}$$

Denote the greatest common divisor of all r-row minors of $G(p)$ by $d_r(p)$. Then $\hat{d}_r(p) \mid d_r(p)$, $(r = 1, 2, \cdots, n)$.

If $d_r(p) \neq \hat{d}_r(p)$,

$$d_r(p) = \eta_r(p)\,\hat{d}_r(p), \qquad (r = 1, 2, \cdots, n), \tag{182}$$

$\eta_r(p)$ being a polynomial of nonzero degree. Consider the r-row minor

$$G\begin{bmatrix} i_1 & i_2 & \cdots & i_{r-1} & i_r \\ n-r+1 & n-r+2 & \cdots & n-1 & n \end{bmatrix}. \tag{183}$$

This minor is formed with the rows numbered $i_1, i_2, \cdots, i_r$ and the last r columns. Let the corresponding minors in $A_*(p)A(p)$ be denoted by

$$K\begin{bmatrix} i_1 & i_2 & \cdots & i_{r-1} & i_r \\ n-r+1 & n-r+2 & \cdots & n-1 & n \end{bmatrix}.$$

From the form of $G(p)$,

$$G\begin{bmatrix} i_1 & i_2 & \cdots & i_{r-1} & i_r \\ n-r+1 & n-r+2 & \cdots & n-1 & n \end{bmatrix}$$
$$= \psi_{i_1}\psi_{i_2} \cdots \psi_{i_r}\psi_{n-r+1*} \cdots \psi_{n*}$$
$$\cdot K\begin{bmatrix} i_1 & i_2 & \cdots & i_{r-1} & i_r \\ n-r+1 & n-r+2 & \cdots & n-1 & n \end{bmatrix}. \tag{184}$$

The right-hand side of (184) must be divisible by $\eta_r(p)\hat{d}_r(p)$ or, by (180), $\eta_r(p)$ must divide

$$\frac{\psi_{i_1}\psi_{i_2} \cdots \psi_{i_r}}{\theta_r(p)}$$
$$\cdot K\begin{bmatrix} i_1 & i_2 & \cdots & i_{r-1} & i_r \\ n-r+1 & n-r+2 & \cdots & n-1 & n \end{bmatrix}. \tag{185}$$

Since $\psi_n \mid \psi_{n-1} \mid \cdots \mid \psi_1$, $\theta_r(p) \mid \psi_{i_1}\psi_{i_2} \cdots \psi_{i_r}$, and this polynomial quotient is strict Hurwitz. Noting that $| G(p) | =$ constant $\times\ \hat{d}_n(p)$, it is clear that any zero of $\eta_r(p)$ must be a zero of $\psi_1(p)\psi_{1*}(p)$. If $\eta_r(p)$ does possess a right-hand zero p_0, $(p - p_0)$ must be a factor of all K's appearing in (185) for everyone of the ${}^nC_r$ choices of $i_1, i_2, \cdots, i_r$. In a similar manner, by arguing with minors formed with the last r rows of $G(p)$, it can be concluded that if p_0 is a left-hand zero of $\eta_r(p)$, the linear factor $(p - p_0)$ must divide the ${}^nC_r$ minors

$$K\begin{bmatrix} n-r+1 & n-r+2, & \cdots, & n-1 & n \\ i_1 & i_2 & , \cdots, & i_{r-1} & i_r \end{bmatrix}.$$

Consequently if $\eta_r(p)$ possesses either a left- or right-hand zero p_0, at least one row or column of the rth compound [8] of $A_*(p)A(p)$ is divisible by the linear factor $(p - p_0)$. But this is impossible since any compound of an elementary polynomial matrix is an elementary polynomial matrix.[6] Thus $\eta_r(p) = 1$, $(r = 1, 2, \cdots, n)$, Q.E.D.

[6] If A is an arbitrary $n\times n$ matrix and A_r its rth compound, $|A_r| = |A|^{n-1}C_{r-1}$.

By Theorem 2, there exist two elementary polynomial atrices $P^{-1}(p)$ and $F^{-1}(p)$, such that

$$G(p) = H_*(p)H(p),$$

here

$$H(p) = P^{-1}(p)\Psi(p)F^{-1}(p).$$

hus, the matrix

$$V(p) = A(p)\Psi_*(p)F(p)\Psi^{-1}(p)P(p)$$

paraconjugate unitary and regular and has the S.M. anonic form $D(p)$, Q.E.D.

The fine structure of rational, regular, para-unitary atrices stands completely revealed in the beautiful ormula (178) and is an excellent example of the power f Theorem 2.

There still remain many difficult problems of classifi-ation which the author hopes to discuss in the near ature. Some of these problems have been partially re-olved in Oono and Yasuura, [6] which is, to date, un-oubtedly the outstanding paper on the subject.

V. Conclusions

The purpose of this paper has been to present a readable nd systematic account of the more recent developments oncerning the difficult but important problem of rational actorization of rational matrices, and to illustrate the heory by nontrivial examples. The main result is em-odied in Theorem 2, and it would be extremely useful o have available a computer routine for this very valuable nd fundamental algorithm. The memory requirements re probably too severe for present-day digital computers, ut the possibility should be explored.

Since nonrational matrices can be approximated as losely as desired by rational matrices, Theorem 2 pro-ides, in a sense, an effective solution of the Hilbert problem for the semi-infinite line and the class of positive paraconjugate hermetian matrices [11].

Acknowledgment

The unique work of Oono and Yasuura for which the author has already expressed his great admiration is not only a significant contribution to the literature of network synthesis but also to the algebra of rational matrices, and deserves much more attention than has been accorded to it. If the present paper succeeds in improving this situation and stimulating research in this direction, one of its main objectives will have been realized.

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Correspondence

Noise In An Amplitude Selective Detector*

The second detector of a superheterodyne receiver is commonly referred to as an envelope detector and can be represented as shown in Fig. 1. The graph identifying the nonlinear circuit is called the voltage transfer characteristic in which output and input voltages are plotted vertically and horizontally, respectively.

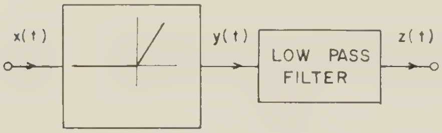

Fig. 1—Block diagram of linear detector.

A great deal of information has been published on the behavior of various forms of envelope detector circuits for various types of input signals and noise. Probably the best known is the work of Rice [1] in which he calculated probability functions of the envelope of a sine wave added to to Gaussian noise. Although the circuit of Fig. 1 performs essentially the function of envelope detection, $z(t)$ is not strictly proportional to the envelope. This linear detector has also been analyzed by a number of writers. Appropriate mathematical techniques and original literature references are given by Davenport and Root, [2] among others.

In this paper, a detector circuit is considered having a band-pass voltage transfer characteristic as shown in Fig. 2. If the circuit of Fig. 2 were used in a radio receiver (for demodulation of pulses, for example), the effect of the band-pass nonlinear element would be to make the detector amplitude selective. That is, the output voltage $z(t)$ would be large only if the RF pulses at the input possessed the appropriate amplitude.

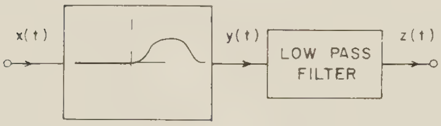

Fig. 2—Amplitude selective detector.

A detector of this type might be used in desensitizing a receiver to impulsive interference caused by electrical discharges or by high-power pulse transmitters located in proximity to the receiver. This technique has been used in radar receivers to a minor extent [3]. Its usefulness is obviously limited to cases in which the interfering pulses are very large relative to information-bearing signals. Basically, however, an amplitude-sensitive detector is useful whenever information is contained in the received strength of pulse signals.

* Received by the PGIT, June 21, 1960; revised manuscript received, November 28, 1960.

Noise in an Amplitude-Sensitive Detector

Two parameters of the random noise at the output $z(t)$ are of interest here: the mean value, m_z, and the variance, σ_z^2. In general these quantities can both be obtained from the input autocorrelation function $R_x(\tau)$. Specifically,

$$\sigma_z^2 = R_z(0) - m_z^2$$

$$R_z(\tau) = \iint h(u)h(v)R_y(\tau + v - u)\, du\, dv.$$

The linear low-pass filters will be assumed to have a dc gain of unity which means that,

$$m_z^2 = m_y^2 = \lim_{\tau\to\infty} R_y(\tau)$$

$$\left.\begin{aligned} R_y(\tau) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} y_1y_2p_y(y_1, y_2; \tau)\, dy_1\, dy_2 \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x_1)f(x_2)p_x(x_1, x_2; \tau)\, dx_1\, dx_2 \end{aligned}\right\}.$$

The above equations give the mathematical machinery necessary to compute m_z and σ_z. Notice that it is sufficient to know: 1) $p_x(x_1, x_2; \tau)$, the second-order probability-density (p.d.) function, corresponding to the input x, 2) $f(x)$, the voltage transfer characteristic of the nonlinear device, and 3) $h(t)$, the impulse response of the linear filter.

$R_y(\tau)$ can be evaluated exactly in terms of a closed expression when the following assumptions are made:

1) $x(t)$ originates from a stationary Gaussian random process;

2) $f(x)$ is Gaussian in shape,

$$f(x) = e^{-(x-A)^2/2a^2}.$$

The following paragraphs are devoted to the special case specified by assumptions 1) and 2).

Since $x(t)$ is from a Gaussian process, $p_x(x_1, x_2; \tau)$ is of the form

$$p_x(x_1, x_2; \tau) = \frac{1}{2\pi\sigma_x^2[1 - \rho_x^2(\tau)]^{1/2}} \cdot \exp\left[\frac{(x_1-A-\Delta)^2 + (x_2-A-\Delta)^2 - 2(x_1-A-\Delta)(x_2-A-\Delta)\rho_x(\tau)}{2\sigma_x^2[1 - \rho_x^2(\tau)]}\right],$$

where $A + \Delta = m_x$.

The exponent of the integrand of $R_y(\tau)$ takes on a quadratic form. By means of the linear transformation,

$$x_1 = v_1 + A + \Delta - k,$$

$$x_2 = v_2 + A + \Delta + k,$$

the exponential integrand may be written,

$$\exp\left\{-\frac{v_1^2 + v_2^2 - 2\rho v_1 v_2}{2\sigma^2(1 - \rho^2)} - \frac{K}{2}\right\}.$$

Since K is independent of the integration,

$$R_y(\tau) = \frac{e^{-K/2}}{\sigma_x^2[1 - \rho_x^2(\tau)]^{1/2}} \cdot \sigma^2[1 - \rho^2]^{1/2}.$$

Like K, σ and ρ are products of the transformation which depend on σ_x, ρ_x, and a. Specifically,

$$\left.\begin{aligned} \sigma^2[1 - \rho^2]^{1/2} &= \frac{\sigma_x^2(1 - \rho_x^2)^{1/2}}{\left[1 + 2\left(\frac{\sigma_x}{a}\right)^2 + \left(\frac{\sigma_x}{a}\right)^4(1 - \rho_x^2(\tau))\right]^{1/2}} \\ K &= \frac{2(\Delta/a)^2}{1 + \left(\frac{\sigma_x}{a}\right)^2[1 + \rho_x(\tau)]} \end{aligned}\right\}.$$

Finally,

$$R_y(\tau) = \frac{\exp\left\{-\dfrac{(\Delta/a)^2}{1 + (\sigma_x/a)^2[1 + \rho_x(\tau)]}\right\}}{\{[1 + (\sigma_x/a)^2]^2 - [(\sigma_x/a)^2\rho_x(\tau)]^2\}^{1/2}}.$$

The above equation holds for a Gaussian-shaped detector characteristic and Gaussian noise.

$$\rho_x(\tau) = \frac{R_x(\tau) - m_x^2}{\sigma_x^2},$$

where $R_x(\tau)$ is the autocorrelation function of the input noise.

Signal-to-Noise Ratio

It is informative to determine the mean and rms values of the output z at the trailing edge of rectangular input signal pulses mixed additively with Gaussian noise. Prior to the trailing edge of a rectangular pulse for a time equal to its duration, the input $x(t)$ is still stationary and Gaussian, satisfying the previous assumption. The value of m_z at an instant corresponding to the trailing edge of the input pulse is defined m_s. σ_s will be similarly defined.

To evaluate the SNR defined later, it is necessary to compute m_z and σ_z when noise only is present. The mean and rms values under these conditions will be referred to as m_N and σ_N, respectively.

If $h(t)$ is assumed to be rectangular and of length T, $[R_z(\tau)]_{\tau=0}$ reduces to the form

$$R_z(0) = \sigma_z^2 + m_z^2 = 2/T \int_0^T (1 - \tau/T) R_y(\tau)\, d\tau,$$

$$= 2/T \int_0^T \frac{(1 - \tau/T) \exp\left\{\dfrac{-(\Delta/a)^2}{1 + (\sigma_x/a)^2[1 + \rho_x(\tau)]}\right\}}{\{[1 + (\sigma_x/a)^2]^2 - (\sigma_x/a)^4 \rho_x^2(\tau)\}^{1/2}}\, d\tau.$$

A closed form for σ_z^2 follows for the case where $\Delta = 0$ and where $\rho_x(\tau) = e^{-2\pi W\tau}$. This corresponds to the condition wherein the peak of the signal pulse is centered with respect to $f(x)$ (Fig. 3).

$$[\sigma_z^2]_{\Delta=0} = \sigma_S^2 \cong \frac{m_S^2}{2\pi WT} \cdot \log\left[4(1 + \alpha^2) \cdot \frac{1 - \sqrt{1 - \left(\dfrac{1}{1 + \alpha^2}\right)^2}}{1 + \sqrt{1 - \left(\dfrac{1}{1 + \alpha^2}\right)^2}}\right]$$

$$\cong \frac{m_S^2}{2\pi WT} \log f(\alpha), \qquad WT > 1,$$

where $\alpha = a/\sigma_x$.[1]

When noise only is present, $m_x = A + \Delta = 0$ and $\Delta = -A$. The value of σ_N can be written approximately,

$$[\sigma_z^2]_{\Delta=-A} \cong \sigma_N = \frac{m_N}{\sqrt{2\pi WT}} \cdot \left[\log f(\alpha) + \frac{2\mathcal{R}^2}{(1 + \alpha^2)^2}\right]^{1/2}.$$

The mean values m_s and m_N can be evaluated from $R_y(\tau)$ as $\tau \to \infty$ as given in:

$$\left.\begin{aligned} m_S^2 &= \frac{\alpha^2}{1 + \alpha^2} \\ m_N^2 &= \frac{\alpha^2}{1 + \alpha^2} e^{-A^2/\sigma_x^2/1+\alpha^2} \end{aligned}\right\}.$$

Fig. 3 illustrates the significance of the output SNR.

$$\mathcal{R}_0 = \frac{m_S - m_N}{\sigma_S + \sigma_N}.$$

$\mathcal{R}_0$ is the distance from V to m_s expressed in σ_s units, or the distance between V and m_N expressed in σ_N units;

$$V \text{ is defined by } \frac{m_S - V}{\sigma_S} = \frac{m_N - V}{\sigma_N}.$$

Hence, $\mathcal{R}_0$ measures the number of standard deviations that a threshold, set for approximately equal signal and false alarm probabilities, is separated from the mean of either signal-plus-noise or noise only distributions.

Substituting m_S, m_N, σ_S, and σ_N into the defining equation, we obtain

$$\frac{\mathcal{R}_0}{\sqrt{2\pi WT}} = \frac{1 - e^{-(\mathcal{R}^2/2)/(1+\alpha^2)}}{[\log f(\alpha)]^{1/2} + e^{-(\mathcal{R}^2/2)/(1+\alpha^2)}\left[\log f(\alpha) + 2\dfrac{\mathcal{R}^2}{(1 + \alpha^2)^2}\right]^{1/2}}.$$

$\mathcal{R}_0/\sqrt{2\pi WT}$ is plotted in Fig. 4 for three values of the input SNR, $\mathcal{R}$, and includes data points obtained from measurements obtained from the nonlinear device described by Figs. 5 and 6 (next page). Fig. 5 shows a rather good fit to the true Gaussian characteristic assumed in the analysis; the circuit of Fig. 6 describes the experimental setup and the diode logic circuit arrangement used to obtain this characteristic ($V_3 = 5$ volts). An RC integrator, in this application, is approximately equivalent to the ideal (assumed in the analysis) as long as $RC \gg 1/2\pi W$.

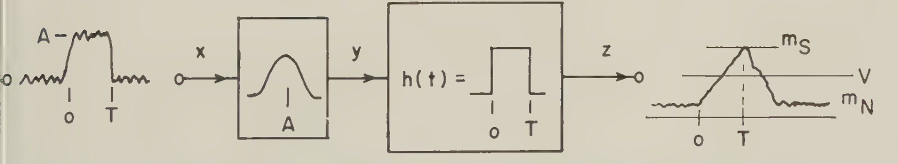

Fig. 3—Input and output waveforms.

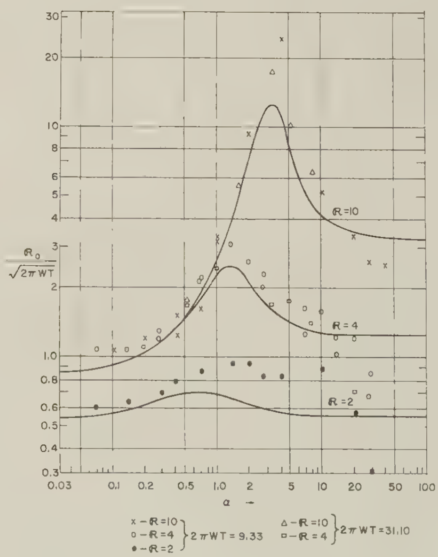

Fig. 4—Normalized output SNR and measured data (all data points lowered by 10 per cent).

[1] An exact closed expression for σ_s^2 is given in [7].

Fig. 5—Gaussian nonlinear characteristic.

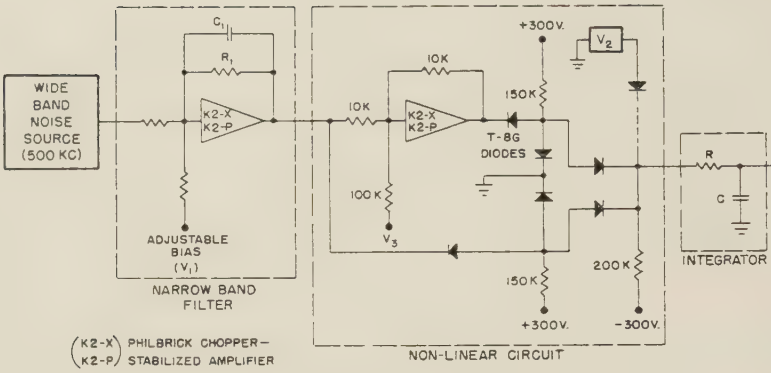

Fig. 6—Schematic diagram of experimental amplitude filter.

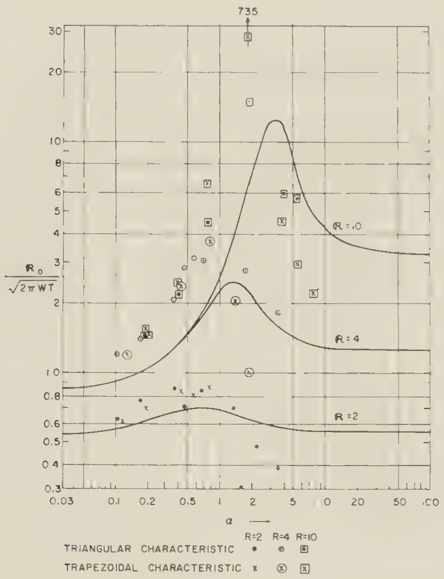

Fig. 7—$\mathcal{R}_0/\sqrt{2\pi WT}$ measured for triangular and trapezoidal characteristics.

amount of noise mixed with the rectangular signal pulse.

Triangular and trapezoidal nonlinear characteristics were checked experimentally and compared with the curves calculated for the Gaussian case Fig. 7. Although the points quite naturally do not agree with the curves, the tendency again is exhibited for the maxima to occur at points indicating a fixed width-to-signal height ratio.

It should be emphasized that this analysis was performed assuming rectangular video pulses mixed additively with Gaussian noise, and hence, does not apply directly to the demodulator example of Fig. 2. However, it is reasonable to expect that the same type of behavior would result; *i.e.*, that there exists for the IF case a fixed ratio of amplitude bandwidth-to-signal height ratio for maximum output SNR.

The results do apply directly to the coherent radio frequency receiver which has available carrier frequency and phase information. The known carrier is used as the reference signal which is combined with the IF in a phase detector pulse demodulation. In this case, the phase detector output could be approximated by rectangular video pulses mixed additively with Gaussian noise.

Conclusions

In the sense that the amplitude sensitive detector (Figs. 2 and 3) is capable of separating a pulse of known height from other larger or smaller pulses (nonoverlapping in time), the circuit can be thought of as a band-pass amplitude filter. With this in mind we ask the question: what amplitude bandwidth maximizes the SNR $\mathcal{R}_0$?

The calculated curves of Fig. 4 show maxima when $a \simeq a_m = A/3$ ($\alpha = a/\sigma_x$, $\mathcal{R} = A/\sigma_x$). Although this was not checked for all $\mathcal{R}$, the curves show that in the range of $2 \leq \mathcal{R} \leq 10$, that the best value of a, does not depend strongly on the

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A Frequency-Weighted Mean-Square Error Criterion*

The mean-square error criterion is commonly used for the optimization of linear filters which perform predicting and smoothing operations upon random processes. The mean-square error may be expressed in the following form:

$$EMS = \int_{-\infty}^{+\infty} d\lambda f_e(\lambda), \qquad (1)$$

where $f_e(\lambda)$ is the power spectral density of the error. The error power spectrum is a function of the signal and noise power spectra, the amount of delay or prediction desired, and the frequency function of the filter. Expression (1) is minimized by adjusting the filter function.

To write $f_e(\lambda)$, the following nomenclature is used:

$$s(t) = \text{signal}.$$

$$n(t) = \text{noise}.$$

Ensemble averages are represented by $E[\;]$.

$\psi_{sn}(\tau) = E[s(t)n\,(t+\tau)]$ = cross-correlation function.

$$f_{sn}(\lambda) = \int_{-\infty}^{+\infty} d\tau \psi_{sn}(\tau) \exp\left[-2\pi i \lambda \tau\right]$$

= cross-spectral density.

$f_s(\lambda)$ = signal power spectrum.
$f_n(\lambda)$ = noise power spectrum.
$k(\lambda)$ = filter frequency function.
λ = frequency in cps.
D = delay time.

The signal-plus-noise spectrum is denoted by

$$g(\lambda) = f_s(\lambda) + f_{sn}(\lambda) + f_{ns}(\lambda) + f_n(\lambda).$$

It is assumed that $f_s(\lambda)$, $f_n(\lambda)$, and $g(\lambda)$ have the Hopf-Wiener factorization $f(\lambda) = |f_-(\lambda)|^2 = f_-(\lambda)\, f_+(\lambda)$. $f_-(\lambda)$ has poles and zeros only in the upper half plane. $f_+(\lambda)$ has poles and zeros only in the lower half plane.

The input to the filter is $s(t) + n(t)$. The filter output is denoted by $\phi(t)$. The

* Received by the PGIT, September 28, 1960; revised manuscript received, December 9, 1960. The work reported here was supported by Bell Telephone Labs., Murray Hill, N. J.

ror is represented by $\phi(t) - s(t - D)$. hus the error power spectrum is

$$(\lambda) = g(\lambda) \mid k(\lambda) \mid^2 - 2 \operatorname{Re} \{[f_s(\lambda) + f_{sn}(\lambda)]k(\lambda) \cdot \exp [2\pi i \lambda D]\} + f_s(\lambda). \quad (2)$$

In many instances the error power is itical for only certain frequency ranges. lthough error power outside those ranges ay be large, its effect may not be disurbing. Multiplying the error power pectrum by a frequency function which is rge for the critical frequency ranges and nall elsewhere will yield a weighted meanuare error. The frequency-weighting inction is designated by $C(\lambda)$ and it is ssumed to be Hopf-Wiener factorable. he weighted mean-square error may be ritten as

$$EMS_w = \int_{-\infty}^{+\infty} d\lambda f_e(\lambda)C(\lambda). \quad (3)$$

The filter that minimizes EMS_w is adily obtained from Wiener's work.[1] The sult is

$$_w(\lambda) = \frac{1}{g_-(\lambda)C_-(\lambda)} \cdot \int_0^{\infty} dt \exp [-2\pi i \lambda t] \cdot \int_{-\infty}^{+\infty} du \frac{[f_s(u) + f_{ns}(u)]C_-(u)}{g_+(u)} \cdot \exp [2\pi i u(t - D)]. \quad (4)$$

In using a frequency-weighted criterion, re must be taken so that the output of e optimal filter will be finite. If $C(\lambda)$ is λ^n), $n \geq 0$, as $\lambda \to \infty$, then the output ower of the filter will be finite. Thus, in eneral, the weighting function may be very nall at high frequencies, but it cannot go zero as λ approaches infinity unless a nite output power restriction is explicitly nposed upon the system. Should $C(\lambda) \to 0$ $\lambda \to \infty$ and there is no explicit finite ower restriction, then the integral of the itput power spectrum of the optimal filter ill diverge.

If the integral diverges, a finite power striction may be imposed by means of agrange's method of undetermined multiiers. The filter output is specified by

$$P_0 = \int_{-\infty}^{+\infty} d\lambda \mid k(\lambda) \mid^2 g(\lambda). \quad (5)$$

$_0$ is required to be equal to some constant, ch as the saturation level of the system. he optimal filter must minimize

$$\int_{-\infty}^{+\infty} d\lambda f_e(\lambda)C(\lambda) + \gamma \int_{-\infty}^{+\infty} d\lambda \mid k(\lambda) \mid^2 g(\lambda), \quad (6)$$

where γ is some constant, not immediately specified. To obtain the optimal filter, define $g_\gamma(\lambda) = g(\lambda) [C(\lambda) + \gamma]$. From a comparison of expressions (2)–(4) with (6), the optimal filter is

$$k_w(\lambda; \gamma) = \frac{1}{g_{\gamma-}(\lambda)} \cdot \int_0^{\infty} dt \exp [-2\pi i \lambda t] \cdot \int_{-\infty}^{+\infty} du \frac{[f_s(u) + f_{ns}(u)]C(\lambda)}{g_{\gamma+}(u)} \cdot \exp [2\pi i u(t - D)]. \quad (7)$$

γ is adjusted so that (6) is satisfied.

The level and form of the weighting function in the noncritical regions is of importance in determining the optimal filter. Intuitively, little importance would be attached to the precise value of the weighting function in such regions as long as the function is sufficiently small. However, the effect of frequency weighting is to cause the filter to attenuate, relative to the unweighted criterion, the error power in the critical ranges. In doing so, the error power is increased in the noncritical ranges. The amount of additional error power that can be tolerated in the noncritical ranges specifies the weighting function in that range. Crudely, there is an inverse relationship between the error power and weighting function at each frequency. Thus, the value of the weighting function is critical in the frequency ranges where the error is noncritical.

ACKNOWLEDGMENT

The author wishes to thank Prof. P. M. Schultheiss for his comments and encouragement.

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Information Theory and the Separability of Signals with Overlapping Spectra*

The author[1] has recently applied the noiseless coding theorem of information theory to obtain a result which is, in some respects, a generalization of the sampling theorem. The technique employed was to replace the probability function of the coding theorem by a quantity proportional to the spectral function of a random process. It may be of some interest to see that the coding theorem for a noisy channel can be adapted in a similar fashion.

Mathematically speaking, the principal theorems of information theory are asymptotic statements about probability measures. Thus, the theorems, when interpreted appropriately, yield statements about any functions which have the same properties as probability density or distribution functions.

Let $x_1(t)$, $x_2(t)$, $\cdots$, $x_M(t)$ be M uncorrelated stationary random processes with possibly overlapping spectra. The main object of this note is to obtain a measure of the amount of overlap of these spectra. This measure is obtained by determining how many of the products $x_{k_1}(t_1)\, x_{k_2}(t_2) \cdots x_{k_n}(t_n)$ can be separated by filtering in n-dimensional space when n is large. Except for its conceptual value in providing a measure of overlap of spectra, no application of the result is apparent to the writer. However, it is conceivable that signals which are functions of n time variables will be used in the future.

Let the processes have mean zero, variance one, autocorrelation functions $r_k(t)$, and spectral density functions $S_k(f)$ for $k = 1, 2, \cdots, M$. That is,

$$E[x_k(t)x_k(t + \tau)] = r_k(\tau), \quad (1)$$

$$\int_{-\infty}^{\infty} r_k(\tau)e^{-2\pi i f \tau}\, d\tau = S_k(f), \quad (2)$$

and

$$\int_{-\infty}^{\infty} S_k(f)\, df = 1. \quad (3)$$

Now let $x_k^{(j)}(t)$ ($j = 1, 2, \cdots, n$) be n uncorrelated random processes, all having the same spectral density $S_k(f)$. Let u_i denote the sequence of integers $\{k_1, k_2, \cdots, k_n\}$ where $1 \leq k_j \leq M$. There are M^n such sequences u_i. Finally, let

$$y(u_i; t_1, t_2, \cdots, t_n) = x_{k_1}^{(1)}(t_1)x_{k_2}^{(2)}(t_2) \cdots x_{k_n}^{(n)}(t_n). \quad (4)$$

Then the autocorrelation function of y is given by

$$\rho(u_i; \tau_1, \cdots, \tau_n) = E[y(u_i; t_1, \cdots, t_n)y \cdot (u_i; t_1 + \tau_1, \cdots, t_n + \tau_n)] = r_{k_1}(\tau_1) \cdots r_{k_n}(\tau_n). \quad (5)$$

Similarly, the spectral density function of y is given by

$$S(u_i; f_1, \cdots, f_n) = S_{k_1}(f_1)S_{k_2}(f_2) \cdots S_{k_n}(f_n). \quad (6)$$

It is the functions $y(u_i, t_1, \cdots, t_n)$ rather than the individual functions $x_k(t)$ which, for suitably large n, will be separated by linear filtering.

* Received by the PGIT, October 19, 1960; revised manuscript received, January 24, 1961.
[1] L. L. Campbell, "Minimum coefficient rate for stationary random processes," *Inform. and Control*, vol. 3, pp. 360–371; December, 1960.

It is now possible to consider a noisy semicontinuous channel without memory which has the preceding spectral density functions as its probability functions. The space of input signals will be the set of integers 1,2, $\cdots$, M and the space of received signals will be the real line $-\infty < f < \infty$. In view of (3), $S_k(f)$ can be regarded as a conditional probability density function for a received signal f, given the input signal k. Similarly, the extension of length of n of this channel has for input space the M^n sequences $u_i = \{k_1, k_2, \cdots, k_n\}$ and for received signal space the n-dimensional space $-\infty < f_j < \infty (j = 1,2, \cdots, n)$. The conditional probability density function is $S_{k_1}(f_1) S_{k_2}(f_2) \cdots S_{k_n}(f_n)$.

Let p_k ($k = 1, 2, \cdots, M$) be M non-negative numbers satisfying

$$\sum_{k=1}^{M} p_k = 1, \tag{7}$$

and let $S(f)$ be defined by

$$S(f) = \sum_{k=1}^{M} p_k S_k(f). \tag{8}$$

Then the capacity, C, of this channel is given by

$$C = \max [H(X) - H(X \mid \Omega)], \tag{9}$$

where

$$H(X) = -\sum_{k=1}^{M} p_k \log p_k, \tag{10}$$

$$H(X \mid \Omega) = -\sum_{k=1}^{M} \int_{-\infty}^{\infty} \log \left[\frac{p_k S_k(f)}{S(f)}\right] p_k S_k(f)\, df, \tag{11}$$

and the maximum is taken over all possible sets of values of $p_1, \cdots, p_M$ which satisfy (7). Since there is no special advantage in using logarithms to the base two in this problem, all logarithms in this paper will be natural logarithms.

The coding theorem then states the following:[2] Let H and ϵ be two numbers satisfying $0 < H < C$ and $\epsilon > 0$. Then there exists a positive constant n_0 such that in every extension of length $n \geq n_0$ there exists a set $u_1, u_2, \cdots, u_N$, $N \geq e^{nH}$, to each of which is associated a set A_i ($i = 1,2, \cdots, N$) such that $P(A_i|u_i) \geq 1 - \epsilon$. Moreover, the sets A_i are disjoint. Here the sets u_i are N sequences from among the M^n sequences $\{k_1, k_2, \cdots, k_n\}$. Each set A_i is a set of points in the n-dimensional space $-\infty < f_j < \infty$ ($j = 1,2, \cdots, n$). The probability $P(A_i|u_i)$ is the conditional probability of the received signal lying in A_i when the input is the sequence u_i.

The coding theorem must now be translated back to spectral terminology. The theorem states that there are N sets of integers $\{k_1, k_2, \cdots, k_n\}$ and N disjoint sets A_i in n-dimensional space such that

$$\int_{A_i} \cdots \int S_{k_1}(f_1) \cdots S_{k_n}(f_n) \cdot df_1 \cdots df_n \geq 1 - \epsilon. \tag{12}$$

Now the integrand is just the spectral density function of the function $y(u_i; t_1, \cdots, t_n)$ defined by (4). Thus (12) says that a fraction $1 - \epsilon$ of the power in $y(u_i; t_1, \cdots, t_n)$ falls in the set A_i for $i = 1,2, \cdots, N$. Moreover, no more than a fraction ϵ of the power from any of the other functions $y(u_j; t_1, \cdots, t_n)$ falls in A_i for $j \neq i$ and $j = 1,2, \cdots, N$. Therefore, since ϵ was arbitrary, and since the sets A_i are disjoint, it is possible to obtain arbitrarily good separation of N signals $y(u_i; t_1, \cdots, t_n)$ by a "filtering" process in n-dimensional frequency space.

Somewhat more roughly, there are e^{nC} distinguishable products $x_{k_1}{}^{(1)}(t_1) \cdots x_{k_n}{}^{(n)}(t_n)$. Since the number of possible products is M^n, it is natural to take a geometric mean and say that there are e^C distinguishable random processes in the original set $x_1(t), \cdots, x_M(t)$. Of course, the term "distinguishable" must be understood only in the sense used here.

In the trivial case that the spectra of the $x_k(t)$ are nonoverlapping, *i.e.*, that $S_j(f) S_k(f) = 0$ for all f whenever $j \neq k$, it is easily shown from information-theoretic considerations that $H(X|\Omega) = 0$ and hence that $C = \log M$. In this case there are $e^C = M$ distinguishable functions, in agreement with simpler notions of distinguishability. Similarly, if all the spectra are identical so that $S_k(f) = S(f)$, it follows that $C = 0$. Thus there is only one distinguishable function.

A more interesting example is provided by two partially overlapping rectangular spectra. Let

$$S_k(f) = \begin{cases} (2W)^{-1} & \text{for } -(f_k + W) < f < -f_k \text{ and } f_k < f < f_k + W \\ & (k = 1, 2) \\ 0 & \text{otherwise,} \end{cases} \tag{13}$$

and let $f_1 < f_2 < f_1 + W$. A simple calculation shows that

$$C = \frac{f_2 - f_1}{W} \log 2, \tag{14}$$

and the number of distinguishable signals is

$$e^C = 2^{(f_2 - f_1)/W}. \tag{15}$$

As f_2 increases from f_1 to $f_1 + W$, e^C increases from one to two.

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[2] A. Feinstein, "Foundations of Information Theory," McGraw-Hill Book Co., Inc., New York, N. Y.; 1958.

On the Approximation to Likelihood Ratio Detectors Laws (The Threshold Case)*

In a recent note[1] Bussgang and Mudgett emphasize the fact that for the case of a sine wave in noise it is not sufficient to approximate the logarithm of the likelihood ratio by using only one term in the expansion of $\log I_0(\eta r)$, but that a third term is required in order that the expected value of the detector output will converge with respect to the null hypothesis. In the case of a sequential test two terms in the likelihood ratio lead to an average sample number which diverges at the null hypothesis. I agree completely with the authors that this point, although emphasized in other publications, still is not recognized by many.

Blasbalg[2] has shown that the use of the first term in the approximation to the logarithm of the likelihood ratio always leads to an expected value which is zero under the null hypothesis (at least where the indicated expansion is valid), and hence a divergent ASN Function at this point. We will prove this again.

Assume that we are testing the hypothesis H_0 that the $\theta \leq \theta_0$ against the alternative hypothesis H_1 that $\theta \geq \theta_1 (\theta_0 < \theta_1)$. Then when $\theta_1 - \theta_0 < 1$, the threshold case, the log-likelihood ratio for a single sample is

$$\log \frac{P(r, \theta_1)}{P(r, \theta_0)} = \left(\frac{P(r, \theta_1)}{P(r, \theta_0)} - 1\right) - \frac{1}{2}\left(\frac{P(r, \theta_1)}{P(r, \theta_0)} - 1\right)^2 + \cdots. \tag{1}$$

* Received by the PGIT, November 3, 1960.

[1] J. J. Bussgang and W. L. Mudgett, "A note of caution on the square-law approximation to an optimum detector," IRE Trans. on Information Theory, vol. IT-6, (Correspondence), pp. 504–505; September, 1960.

[2] H. Blasbalg, "The sequential detection of a sine-wave carrier of arbitrary duty ratio in Gaussian noise," IRE Trans. on Information Theory, vol. IT-3, pp. 248–256; December, 1957.

we consider only the first two terms in expansion we have for the detector law

$$= \left(\frac{P(r, \theta_1)}{P(r, \theta_0)} - 1\right) - \frac{1}{2}\left(\frac{P(r, \theta_1)}{P(r, \theta_0)} - 1\right)^2, \qquad (2)$$

d

$$(z) = \int_{-\infty}^{+\infty} \frac{P(r, \theta_1)}{P(r, \theta_0)} \cdot P(r, \theta_0)\, dr - \int_{-\infty}^{+\infty} P(r, \theta_0)\, dr$$

$$- \frac{1}{2}\int_{-\infty}^{+\infty} \left[\frac{P(r, \theta_1) - P(r, \theta_0)}{P(r, \theta_0)}\right]^2 P(r, \theta_0)\, dr$$

$$= 0 - \frac{1}{2}\int_{-\infty}^{+\infty} \left[\frac{P(r, \theta_1) - P(r, \theta_0)}{P(r, \theta_0)}\right]^2 P(r, \theta_0)\, dr$$

$$= -\frac{1}{2} E_{\theta_0}\left[\frac{P(r, \theta_1) - P(r, \theta_0)}{P(r, \theta_0)}\right]^2. \qquad (3)$$

nce, the expected value of the first term nishes under the null hypothesis $\theta = \theta_0$. Let us now obtain these results in a more ognizable form. Let $\theta_1 = \theta_0 + \Delta\theta$ ere $\Delta\theta < 1$. Then,

$$r; \theta_0 + \Delta\theta) = P(r, \theta_0) + \frac{\partial}{\partial\theta} P(r, \theta)\Big|_{\theta=\theta_0} \Delta\theta + \cdots. \qquad (4)$$

we include only the first two terms shown d divide through by $P(r, \theta_0)$, we have

$$\frac{r, \theta_0 + \Delta\theta) - P(r, \theta_0)}{P(r, \theta_0)} = \frac{\Delta\theta}{P(r, \theta_0)} \frac{\partial}{\partial\theta} P(r, \theta)\Big|_{\theta=\theta_0}. \qquad (5)$$

bstituting into (2) for the detector yields

$$= \frac{\Delta\theta}{P(r, \theta_0)} \frac{\partial}{\partial\theta} P(r, \theta)\Big|_{\theta=\theta_0} - \frac{1}{2} \frac{\Delta\theta^2}{[P(r, \theta_0)]^2}\left[\frac{\partial}{\partial\theta} P(r, \theta_0)\right]^2. \qquad (6)$$

It should also be clear that

$$z = \Delta\theta \frac{\partial}{\partial\theta} \log P(r, \theta)\Big|_{\theta=\theta_0} - \frac{1}{2}\Delta\theta^2\left[\frac{\partial}{\partial\theta} \log P(r, \theta)\Big|_{\theta=\theta_0}\right]^2. \qquad (7)$$

If we now take the expected value of (6), we have

$$E_{\theta_0}(z) = \Delta\theta \cdot \int_{-\infty}^{+\infty} \left[\frac{\partial}{\partial\theta} P(r, \theta)\Big|_{\theta=\theta_0}\right] dr - \frac{\Delta\theta^2}{2} \cdot \int_{-\infty}^{+\infty} \left[\frac{\frac{\partial}{\partial\theta} P(r, \theta)\Big|_{\theta=\theta_0}}{P(r, \theta_0)}\right]^2 P(r, \theta_0)\, dr. \qquad (8)$$

Now, if we assume that the order of differentiation and integration can be interchanged as will almost always be the case, then the expected value of the first term in (8) is

$$\Delta\theta \int_{-\infty}^{+\infty} \frac{\partial}{\partial\theta} P(r, \theta)\Big|_{\theta=\theta_0} dr = \Delta\theta\left[\frac{\partial}{\partial\theta}\int_{-\infty}^{+\infty} P(r, \theta)\, dr\Big|_{\theta=\theta_0}\right] = \frac{\partial}{\partial\theta}(1) = 0. \qquad (9)$$

Then, from (7) and (9) we have

$$E_{\theta_0}(z) = -\frac{(\theta_1 - \theta_0)^2}{2} E_{\theta_0} \cdot \left[\frac{\partial}{\partial\theta} \log P(r, \theta)\Big|_{\theta=\theta_0}\right]^2, \qquad (10)$$

where $\Delta\theta = \theta_1 - \theta_0$. Eq. (10) is a well-known result;[3] it is the variance of a maximum likelihood ratio estimate.

In the case of the log $I_0(\eta r)$ detector when we perform the power series approximation and include the fourth-order term to obtain convergence for the expected value at $\theta = \theta_0$ we are in fact computing (6) for the detector law and (10) for its average output. Our conclusion is, therefore, that for threshold parameter detection $\theta_1 - \theta_0 = \Delta\theta < 1$, the detector at least must have the first two terms shown. (Although the significance of this result comes to our attention in sequential detection, we must conjecture that it is just as significant for fixed sample size tests since the results derived represent fundamental properties of likelihood ratio tests in general.)

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[3] A. M. Mood, "Introduction to the Theory of Statistics," McGraw-Hill Book Co., Inc., New York, N. Y.; 1950.

Contributors

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From 1951 to 1957, he was a physicist at the Naval Ordnance Laboratory, Silver Spring, Md., working in fluid dynamics, acoustics, applied probability, and stochastic processes. Since 1957, he has been an associate professor of electrical engineering at Purdue University, Lafayette, Ind. During the summer of 1958 and 1959, he worked for Lincoln Laboratory, Lexington, Mass., and in 1960, at Bell Telephone Laboratories, Murray Hill, N. J.

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From 1956–1960, he was engaged in communication and radar research studies at Hughes Aircraft Company, Culver City, Calif. During this time, he also taught part-time at the University of Southern California, Los Angeles, and at the University of California at Los Angeles. He is presently visiting lecturer in electrical engineering at the University of California, Berkeley, while on leave of absence from Hughes.

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Solomon W. Golomb was born in Baltimore, Md., on May 31, 1932. He received the B.A. degree in mathematics from The Johns Hopkins University, Baltimore, in 1951, and the M.A. degree from Harvard University, Cambridge, Mass., in 1953. After spending the academic year 1955–1956 in Oslo, Norway, on a Fulbright Grant, he received the Ph.D. degree from Harvard in 1957.

He joined the staff of the Jet Propulsion Laboratory at the California Institute of Technology, Pasadena, in 1956, and is presently the Assistant Section Chief of its Communication Systems Research Section. He has been concerned primarily with the applications of discrete mathematics to coding and communications.

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Wayne R. Cowell was born in Wakefield, Kans., on June 27, 1926. He received the B.S. and M.S. degrees in mathematics from Kansas State University, Manhattan, in 1948 and 1950, respectively, and the Ph.D. degree in mathematics from the University of Wisconsin, Madison, in 1954.

From 1954 to 1959 he was assistant professor of mathematics at Montana State University, Missoula, where his activities included teaching and research in abstract algebra and projective geometry. Since 1959, he has been with Bell Telephone Laboratories, Inc., Murray Hill, N. J., where he is engaged in basic studies in data communication. His present interests are primarily in coding theory and error-control systems.

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D. C. Youla (SM '59) was born in Brooklyn, N. Y., on October 17, 1925. He received the B.E.E. degree from the College of the City of New York in 1947, and the M.S. degree from New York University in 1950.

From 1947 to 1949 he was employed as an instructor in the Department of Electrical Engineering at C. C. N. Y. He attended the N. Y. U. Graduate School of Mathematics as a full-time student from 1948 to 1950, and for the next two years was at Fort Monmouth, N. J., and Brooklyn Naval Shipyard working on problems of UHF and microphonics. In 1952 he joined the communication group at the Jet Propulsion Laboratories, Pasadena, Calif., and participated in the design of antijam radio links for guided missiles. In 1955 he began his present association with the Microwave Research Institute, Polytechnic Institute of Brooklyn, Brooklyn, N. Y., where he engaged in the practical and theoretical study of codes for combating noise and improving efficiency. He is now a research associate professor of electrical engineering, working actively on network synthesis problems, stability of time-variable systems, solid-state devices and n-port filtering.

bstracts

This Section of the issue is devoted to abstracts of material which may be of interest to PGIT members. Sources are Government, Industrial and University reports, and books and journals published outside of the United States. Readers familiar with material of this nature which is suitable for abstracting are requested to communicate the pertinent information to one of the Editors or Correspondents listed below.

equency-Time Transposition for the Measurement of an Unknown equency, I—R. H. Baumann (in French). (*Ann. de Radioélectricité*, l. 15, pp. 305–330; October, 1960.)

A new method for the determination of the Doppler frequency of inusoidal signal in the presence of noise is described. The system nsists of a delay line in a closed-loop arrangement such that the out signal, whose frequency is to be determined, is allowed to culate several times in the closed-loop system. At each recirculan of the signals in the loop, the frequency of the signals are fted by an amount equal to the reciprocal of the line delay before ey are added to the input signals. These circulating sinusoidal nals are thereby transformed into impulses, whose time shift is a asure of the unknown Doppler frequency. In Part I of this instigation, an idealized system is treated theoretically, and results also given for an experimental system.

versible Stationary Random Functions—A. Blanc-Lapierre (in ench). (*Comptes rend. acad. sci.*, vol. 251, pp. 1957–1959; Novem- 7, 1960.)

The author gives various properties of stationary random funcns whose moments $E[X(t_1) \cdots X(t_n)]$ are invariant when the tants $t_1, \cdots, t_n$ are replaced by $t_1', \cdots, t_n'$, respectively symmetric the former around an arbitrary instant t_0.

Correlator Employing Hall Multipliers Applied to the Analysis Vocoder Control Signals—A. R. Billings and D. J. Lloyd (in glish). (*Proc. IEE*, vol. 107, pt. B, p. 435; September 1960.)

The authors define a *periodic weighted correlation function* which l be obtained when 1) an infinite stationary time function is laced by the cyclic repetition of a time-function of finite length, d 2) the integrators used in the correlator are imperfect. (The ter is accounted for by a weighting function.) The correlator uses Hall effect for multiplication, and signal frequencies of the order 25 cps or less are recorded on magnetic tape in the form of amplile modulation of a 2-kc carrier. Auto and cross correlograms of control signals in a 10-channel vocoder are obtained. Preliminary ults show from auto correlation that the power spectrum of a ntrol signal occupies a bandwidth considerably less than the 25 commonly considered to be necessary, and from cross correlation t there is still considerable redundancy in vocoder signals.

A General Formulation of the Fundamental Theorem of Shannon in the Theory of Information—R. L. Dobrushin (in Russian). (*Uspekhi Matemat. Nauk*, vol. 14, pp. 3–104; November-December, 1959.)

In a valuable book by Shannon and Weaver, the fundamental concepts of a theory of information were introduced, and the fundamental theorem of this theory was obtained at a physical level of rigor. After this the works of MacMillan and Khinchin appeared, in which a strict interpretation of the Shannon theorem was given in the case of a discrete stationary source and channel under the requirement of strict coincidence of the information being received and that being transmitted. In this, Khinchin essentially based himself on the ideas in the work of Feinstein. The works of Khinchin were extended in an application to processes with continuous multiple states by Rosenblat-Roth and more particularly by Perez. Rosenblat-Roth also indicated the possibility of extending the theory to nonstationary processes. Also of interest are the recent works of Wolfowitz and of Blackwell, Breiman and Thomasian.

Kolmogorov, in the form of the organization of the problem, led the way to a highly general and mathematically rigorous treatment of the Shannon theorem. The aim of the present work is to give a proof of Shannon's theorem according to Kolmogorov's interpretation, under sufficiently general conditions. These general conditions are formulated with the help of the concept of information density, introduced into the mathematical literature by Gel'fand and Yaglom, and by Perez. The frequently used less general concept of information stability is evoked, along with some ideas expressed in words by the two first-mentioned authors.

On the Concept of the Instantaneous Frequency of a Signal—R. Fortet (in French). (*Cables et Transmission*, vol. 14, pp. 60–73; January, 1960.)

The paper consists of three parts: the first one presents some general remarks on the definition of the instantaneous frequency of a signal, as derived from the analytic signal concept, and on the corresponding calculation of its value. The second one relates to filters considered as transmitters of infinitely short pulses; its object is to compare the response of the filter to such a pulse to that to a given input signal. In the third part, the author develops and discusses calculation methods applicable to filters of the minimum-phase type, *i.e.*, conforming the Bayard-Bode relation. This study is not complete: only certain basic elements are given, more particularly a theorem according to which a network with a filtering characteristic symmetrical with respect to a central frequency does not cause any instantaneous frequency distortion.

On the Determination of the Amount of Information Concerning a Random Function Supplied by Another Similar Function—I. M. Gel'fand and A. M. Yaglom (in Russian). (*Uspekhi Matemat. Nauk*, vol. 12, pp. 3–52; January, 1959.)

The paper is divided into two chapters and is devoted to the main problem of information theory, that of finding the amount of information $I(\xi, \eta)$ of one random object ξ about another one η. In the first chapter the amount of information is defined and its properties are discussed if ξ and η are of a very general nature, *e.g.*, vectors, functions or generalized functions. The case of vectors is treated in detail starting from the bases given in Appendix 7 of Shannon and Weaver's book. Under rather general set-theoretical assumptions, the theorem is proved that the problem of finding the amount of information may be reduced to that of computation of the Lebesgue-Stieltjes integral

$$\iint P_{\xi\eta}(dx\, dy) \log \frac{P_{\xi\eta}(dx\, dy)}{P_\xi(dx)P_\eta(dy)}.$$

The remainder of the first chapter contains the definition of the amount of information for a wide class of generalized random functions as well as the discussion of its properties.

The second chapter begins with the determination of amount of information for generalized Gaussian random functions. Next, a very elegant formula is derived for the case when ξ and η are vectors. Let $\xi = (\xi_1, \xi_2, \cdots, \xi_k)$, and $\eta = (\xi_{k+1}, \xi_{k+2}, \cdots, \xi_{k+l})$ denote multivariate Gaussian variables with second central moments $m_{ij} = E[\xi_i - E(\xi_i)][\xi_j - E(\xi_j)]$. Let $A = \det m_{ij}$ for $1 \leq i, j \leq k$; $B = \det m_{ij}$ for $k \leq i, j \leq k + 1$ and $C = \det m_{ij}$ for $1 \leq i, j \leq k + 1$. Then

$$I(\xi, \eta) = \tfrac{1}{2} \log \frac{AB}{C}$$

provided $C \neq 0$. This restriction, however, may be circumvented by a suitably chosen linear transformation of coordinates in the space of vectors ξ and η.

The Diffused Radiation due to Distribution Errors—J. Guittet (in French). (*Rev. Tech. C.F.T.H.*, no. 33, pp. 29–57; October, 1960.)

The imprecision of fabrication of an antenna affects the characteristics of its radiation pattern. The author studies this effect in the case of an antenna with large gain and a low level of secondary lobes.

Orthogonal Codes –H. F. Harmuth (in English). (*Proc. IEE*, vol. 107, pt. C, p. 242; September, 1960.)

An orthogonal code m elements long is one of which the characters may be positive and negative directions of m orthogonal vectors in m-dimensional space. An example is a set of 32 binary characters of 16-digit length, having mutual distances of either 8 or 16 digits. An equivalent orthogonal code can also be constructed from a set of sine and cosine functions of limited duration having 1 to 8 cycles in the character interval; 32 characters result from taking 8 sine plus 8 cosine, doubled for plus and minus. The frequency spectrum of each character is then a sin x/x type of function. Alternatively one could postulate spectra which have sinusoidal distributions within a prescribed bandwidth, and the time functions would then be of sin x/x type, and still orthogonal. Reception would be by synchronous demodulator, and square waves would be transmitted as synchronizing signals. The signal/noise advantage of a 16-element 32-character orthogonal code is calculated, relative to a 5-element 32-character teletype code.

Optimum Combination of Pulse Shape and Filter to Produce a Signal Peak upon a Noise Background—H. S. Heaps (in English). (IEE Monograph No. 407E; October, 1960.)

The author seeks the optimum shape of transmitted pulse, $V_i(t)$, and the transfer function $H(\omega)$ of the best linear filter for detecting this pulse against a background of noise having power spectrum $|\sigma(\omega)|^2$ and after transmission through a system which has a transfer function (due both to medium and to transducers) $T(\omega)$. The pulse is regarded as made up of n samples spaced by τ over a total pulse duration d. Examples are quoted for a noise/transmission relationship of the form $|\sigma(\omega)|^2/|T(\omega)|^2 = \exp(-k^2\omega^2)$; $n = 3$ or 10; and $d/2k = 6$, 9 or 15. For $n = 3$ the optimum shape is found to be a cycle of oscillation roughly equivalent to three pulses in the sequence positive, negative, positive. The ratio of squared peak signal in the output to mean square noise in the output, modified in accordance with the forms of $\sigma(\omega)$ and $T(\omega)$, has a maximum value denoted by λ_0. For a fixed pulse length d, the optimum value of λ_0 increases very rapidly as n increases.

The Transmission of Discrete Information through Periodic and Almost-Periodic Channels—K. Jacobs (in German). (*Math. Ann.*, vol. 137, pp. 125–135; 1959.)

This mathematical paper is an extension of Khinchin's proofs of the theorems of MacMillan, Feinstein and Shannon I and II to almost-periodic (in particular, periodic) channels. The ergodic capacity of such channels is the same as that of its stationary average. An almost-periodic source has the same entropy as its stationary average. Feinstein's theorem and Shannon I are valid for almost-periodic channels. This may have an application in satellite communication.

Adaptive Waveform Recognition—C. V. Jakowatz, *et al.* (in English). GE Res. Lab., Schenectady, N. Y., Rept. No. 60-RL-2435E, May, 1960; Rept. No. 60-RL-2353E (Revised), September, 1960.)

This report describes an adaptive waveform recognition system capable of picking out a randomly occurring signal perturbed by additive noise. This system was constructed in the form of a self-adaptive matched filter that learns with experience to adjust its impulse response so that it automatically forms the inverse for the signal mentioned above. Furthermore, it has the capability of portraying its concept of what it thinks the signal is. Provided the conditions for initiating convergence are met with infinite experience, *i.e.*, time, the adaptive filter will approach the ideal matched filter. In practice, infinite experience cannot be realized, and the adaptive filter is inferior to a predesigned matched filter. Its utility and application are where *a priori* design information is not available or is only partially available.

Two methods of operating an adaptive filter of the above types are theoretically investigated. In the priming method, the first approximation of the filter, *i.e.*, the first step in learning, is the adjustment of the filter so that it is the matched filter of a random sample of its input. If experience indicates that there was no signal in that random sample, then the filter will reject it and make another trial. In nonpriming operation, the first approximation to the desired matched filter is continuously changing in a random fashion. Convergence begins when the changing filter reaches a state in which the signal is a component of the matched filter.

An adaptive filter has been constructed consisting of a 10-tap delay line with a 500-cps cutoff frequency. The gain of any tap is determined by the previous experience of the filter. The memory associated with experience consists of condensers. The arithmetic operations associated with that memory are based upon relay switches. The constructed machine operates in the nonpriming mode. The filter readjusts itself whenever the correlation between the matched filter and the incoming waveform exceeds a given threshold. Both filter properties and threshold value are functions of the past experience of the filter.

Performance curves on the filter are presented and indicate performance as a function of Woodward's R and filter parameters. In general, convergence is rapidly initiated for values of R greater than about 10. It is difficult for a human observer to detect visually or acoustically randomly occurring undefined events with this value of R.

An Extension of N. Wiener's Prediction Theory—J. Kondo (in English). (*J. Operations Res. Soc. Japan*, vol. 2, pp. 124–129; January, 1960.)

N. Wiener has introduced the so-called Wiener-Hopf integral equation of a predictor $K(t)$ for a continuous time series $f(t)$ in his prediction theory. It is noted, however, that we have to use the factorization technique to find $K(t)$ from this equation. It is sometimes very difficult to carry out this technique when the autocorrelation function of $f(t)$ is not expressed in a simple form.

The present paper deals with the prediction of a time series $f(t)$ th another time series $g(t)$, by taking account of the cross correla-on between these two time series. In this case, we have a singular tegral equation of $K(t)$, and can obtain the solution of $K(t)$ in neral, without applying the factorization technique. When we sume $f(t) \equiv g(t)$, the result will reduce to the Wiener case. There-re, this method includes the Wiener prediction theory as a special se.

ne Output Spectral Density of a Detector Operating on a FM CW adar Signal in the Presence of Band-Limited White Noise— Lait and A. J. Hymans (in English). (*IEE* Monograph No. 412E; ctober, 1960.)

In a radar system in which the returned echo is made to beat ith transmitted signal, the output from the detector will include e following components: 1) the desired beat note; 2) the normally etected noise; and 3) a random signal produced by interaction etween the FM wave and the noise. It is assumed that detector itput results from interactions between reference signal and noise id between reference signal and echo, but that interaction between ho and noise is negligible. Using the detector model proposed by awson and Uhlenbeck, noise spectral distributions are deduced oth for the quadratic detector and for the linear detector with nall and large SNR's. In general, the predetector bandwidth should no greater than is needed to pass the echo and reference signal; it for very small targets at short range, an increase in bandwidth ay move some of the noise power away from the part of the ectrum occupied by the signal. In every case, the postdetector andwidth should be kept as small as is consistent with the required formation rate.

ne Indeterminacies of Measurements Using Pulses of Coherent lectromagnetic Energy—R. Madden (in English). (IEE Mono-raph No. 417E; November, 1960.)

An idealized radar transmitter emits a single pulse of wavelength and duration τ at time $t = t_0$. The associated receiving system omprises a paraboloid antenna of diameter $2a$; an array of signal etectors in the focal plane of the antenna; associated with each etector, a bank of filters for determining the spectral analysis of e echo signal; and associated with each filter, a clock for deter-ining the time elapsed between the transmission of the pulse and e arrival of a particular frequency component in the echo. The igular resolving power corresponding to an antenna of this aperture $\Delta\varphi \simeq \lambda/2a$, but in order to achieve this, the whole aperature must illuminated simultaneously. In general, this requires a pulse ngth τ such that $c\tau \geq 2a$. However, the range resolution ΔR is order $\frac{1}{2}c\tau$, so that $\Delta\varphi \cdot \Delta R \simeq \lambda_t/2$. Similarly the accuracy with hich radial velocity V_r can be found (by Doppler effect) depends the mean frequency and the duration of the pulse, and it is iown that $\Delta R \cdot \Delta V_r \simeq \lambda_t c/4$. Tangential velocity causes the signal move from one detector to the next, but unless the signal dwells r the full time τ on each detector, the accuracy of determination radial velocity will suffer. Similarly, radial acceleration causes e signal frequency to change and so limits the accuracy of deter-ination of radial velocity. The use of nonsimultaneous measure-ents (*e.g.*, pulse trains or modulated-wave systems) produces nbiguities such as $R_{\text{amb}} \cdot V_{\text{amb}} = \lambda_t c/4$.

etermination of the Structure of a Majority-Decision Element by e Method of Linear Programming—S. Muroga, *et al.* (in Japanese). *. Inst. Elec. Commun. Engrgs. Japan*, vol. 43, pp. 1408–1416; ecember, 1960.)

A majority-decision element is an element in which a finite number inputs, having weights (coupling numbers), are coupled with one itput. The output value is one or zero, and is decided by the ajority decision depending on the coupling numbers. The number Boolean functions which can be realized by a single majority-ecision element is rather small. Thus, it is necessary to determine e category of such Boolean functions (majority-decision function). sing the method of linear programming, we have developed a iterion concerning whether a given Boolean function can or cannot realized by a single majority-decision element, and this method etermines also the most economical structure (coupling numbers and threshold) of a majority-decision element realizing the function. In the formulation of linear programming, the number of constraints is considerably reduced by the properties of majority-decision function. A table is given of majority-decision functions of five or less variables and the structure of majority-decision elements; these are calculated by the above method.

On the Noise Figure of Low-Gain Stages of Amplifiers and its Measurements—T. Namekawa (in Japanese). (*J. Inst. Elec. Commun. Engrs. Japan*, vol. 43, pp. 1329–1334; November, 1960.)

The theory of noise figures has been known for many years. In many cases, the first stages of amplifiers are designed to get better noise performance, and the noise figure has been used as a criterion. It is not sufficient to take noise figure only, when the power gains are comparatively small. Power gain must be taken into considera-tion besides the noise figure. The author has developed here a definition "Iterative Noise Figure" F_i:

$$F_i = 1 + \frac{F - 1}{1 - \frac{1}{G}}.$$

This is useful for determining the noise performance of low-gain first stages, and the main part of F_i is same as the "Noise Measure" M which has been developed by Haus and Adler. The methods of measuring the Iterative Noise Figure or the Noise Measure are discussed. It is possible to determine the values of F_i or M by direct reading from the measurement of one stage under test.

Prediction Theory and Dynamic Programming, II—T. Odanaka (in English). (*J. Operations Res. Soc. Japan*, vol. 3, pp. 88–92; October, 1960.)

The theory of prediction given in this paper is an extension of the previous paper presented at the International Statistical Insti-tute, 32nd Session, 34, 1959. In the previous paper, we were con-cerned with the problem of separating a message from a signal, the message being represented by a discrete time sequence described statistically by a given autocorrelation function; and the signal being represented by still another sequence with a given auto-correlation function and a cross-correlation function with respect to the message. This paper presents some application of the func-tional equation technique of Dynamic Programming to the numerical method of this extended prediction theory.

Some Remarks on the Capacity of a Communication Channel—M. Sakaguchi (in English). (*J. Operations Res. Soc. Japan*, vol. 3, pp. 124–132; January, 1961.)

The transmission of information requires the presence of a source of information coupled with an appropriate channel. An information system is described in terms of joint probabilities of inputs and outputs, and a channel is defined by its transition probabilities. The author discusses a close connection between the capacity theorem and the matching theorem. This paper presents a general theorem which includes these two theorems as the two special cases. An interpretation of capacity is given by introducing cost considerations into the information system.

Shift Registers Generating Maximum-Length Sequences—P. H. R. Scholefield (in English). (*Electronic Tech.*, vol. 37, p. 389; October, 1960.)

A cyclic sequence which contains all possible combinations of binary digits may be used as a source of pseudo-random numbers, or for the generation of digital codes. Such a sequence of s digits may be described by a recurrence formula $a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots c_sa_{n-s}$, and it may be generated by means of a shift register with back connections from various stages to the imput. In a binary system $c = 0$ or 1. $F(x) = 1 - \sum c_ix^i$ is called the *characteristic polynomial* and the condition for the register to generate a full sequence of length n is that $F(x)$ contains no factors and does not divide into x^{k+1} for any k less than $2^n - 1$. It is then shown to be advantageous to replace a single register with multiple taps gener-

ating the feedback to the input by several registers of the same aggregate length. Taps on the first partial register are combined to give an input to the second, etc., and taps on the last provide an input to the first. Examples show the saving in logical circuits which can be secured by this modification.

On the Sampling Theorem of the Second Kind—H. Wolter (in German). (*Arch. Elekt. Übertragung*, vol. 13, pp. 477–484; 1959.)

In an earlier paper, the author has shown that if an object function of finite extension G is imaged by an optical information channel of the aperture 2 W, one can get more than 2 WG information data from it. In this paper, he asks whether it is possible to get more than 2 WG' information data from an image of extension G' and again gives a positive answer. He does not deny the Whittaker Interpolation Theorem, but shows that it is easy to derive wrong conclusions from it.

A Simple Speech Synthesizer—D. J. Wollons and A. M. R. Gill (in English). (*Electronic Tech.*, vol. 37, p. 373; October, 1960.)

Intelligible speech is synthesized on the basis of reproducing two format frequencies, in the ranges 200–1200 cps and 1000–2400 cps, respectively, with fricative excitation. The format frequencies are controlled by two resonant circuits of which the damping is reduced by a valve amplifier and the frequency is controlled by using a reverse-biased diode junction as the tuning capacitor. These resonant circuits are excited by an electrical noise source.

On a two-coordinate plot, normal sounds are represented by points (indicating the values of two format frequencies) and diphthongs are represented by trajectories on the two-coordinate plot. The synthesizer is provided with a joy-stick control for varying simultaneously the two formant frequencies. Intelligibility tests showed approximately 70 per cent correct identification for synthesized single sounds and nearly 100 per cent for words, short phrases and sentences.

The following papers were published singly by the Professional Group on Information Theory (I) and the Professional Group on Automata and Automatic Control (A) of the Institute of Electrical Communication Engineers of Japan, 2–8, Fujimicho, Chiyodaku, Tokyo, Japan. All are in Japanese, except as noted; English abstracts are given when available.

Topological Considerations in Information Recognition (I; October 21, 1960)—H. Enomoto.

In this paper, the topological characteristics of a connecting relation of information are considered. It is proved that the space having the same characteristics as the connecting relation of information is topologically homeomorphic with a multihole torus. Some topological considerations are applied to the information recognition process.

A Few Considerations on Pattern Recognition (A; December 8, 1960)—Y. Iijima.

A Computational Method for Speech Recognition (A; September 8, 1960)—S. Inomata (in English).

A computational program for stationary vowel recognition is proposed; it is called SNCS (Speech Normalizing and Comparing Scheme). In this, the first gestalt properties of the input speech, such as amplitude, time origin, time scale factor and phase distortion, are normalized by a normalizing program composed of Fourier and S-transforms. "Active recognition" of the input speech is done by the comparison of the normalized input speech with similarly normalized kernel speech generated by a speech-generating program. In the course of this comparison operation, the second gestalt properties of speech, such as differences of individual pronunciation and of male and female speech, are completely normalized. A special program, developed to normalize the stationary vowel with respect to its duration, is also incorporated. The distinctive features of this speech recognition program are its "developing" and "statistical" learning abilities.

Generation of Speech by a Digital Computer (I; September 30, 1960)—S. Inomata, *et al.*

A digital computer has been successfully programmed to generate five stationary Japanese vowels. Speech waves have been generated by the approximate evaluation of the fahltung-type integral describing the human speech generating process by means of the simple weighted-sum method. Consideration is also given to the extension of this program to both consonants and nonstationary vowels.

Modification of a Speech-Normalizing Algorithm (I; October 21, 1960)—S. Inomata.

Four modifications of the speech-normalizing algorithm involved in the author's SNCS scheme (see above) have been proposed; these can be executed on a digital computer somewhat faster than the original one. The computation time and accuracy of each modification are discussed.

Synthesis of Speech-Recognizing Algorithm with Learning Abilities—Application of the Golf Method (I; November 11, 1960)—S. Inomata.

Consideration is given to the incorporation of learning processes into the SNCS speech-recognizing algorithm (see above). Higher-order learning processes, in which the mode of operation is changed, are excluded from this first approach. The so-called "inner parameter space" is described, and the learning process is formulated as a difficult problem in nonlinear programming. In order to solve this problem, a powerful method of nonlinear programming, the "Golf Method," is proposed and applied.

A Vocoder for Voice Research (A; September 8, 1960)—S. Inoue.

A Topological Approach to the Construction of Group Codes (I; January 17, 1961)—T. Kasami.

This paper presents a systematic procedure for finding quasi-perfect group codes with given m and d, where m is the number of parity-check digits and d is the nearest-neighbor distance. This procedure may be suitable for digital-computer programming. In particular, in the case where $d \leq 5$, at least one quasi-perfect group code can be obtained through the first few steps.

The paper also proposes a topological method of group-code construction which is based on the above-mentioned procedure. For moderate values of m, quasi-perfect codes with $d = 5$ can be obtained rather easily through this method. Four examples are given.

On a Few Problems of Analog-to-Digital Converters (A; January 16, 1961)— O. Kawatori and A. Kitamura.

FM-Like Characteristics of the Fundamental Frequencies of Speech Sounds (I; October 21, 1960)—T. Koshikawa.

Economics of Coding in Parts Manufacturing (I; December 16, 1960)— H. Kubokoya.

On the Precise Measurements of the Difference Between Two Velocities (A; January 16, 1960)—Y. Matsumoto and N. Tatsuta.

A General Statistical Theory of Noise Measurements (I; January 17, 1961)—M. Ōta and M. Nakagami.

This paper describes certain basic theories and properties connected with the measurement of noise through detector circuits.

coding of Japanese Monosyllables (A; January 16, 1961)—Sakai, *et al.*

e Basic Design of Pattern-Recognition Apparatus (A; January 16, 1)—T. Sakai and T. Fukinuki.

plications of Miyakawa's Multidimensional Sampling Theorem—I; September 30, 1960)—K. Sasakawa.

Coding for an Automatic Reading Apparatus (I; January 17, 1)—S. Shirai and H. Sakaguchi.

A description is presented of how to code alphanumerical char-ers read by an automatic reading apparatus. Pulses resulting m scanning letters vertically are distinguished as short, medium ong, according to length. The number of pulses in each scanning counted, and from this, characteristic patterns of the letters obtained. These patterns are distinguished into groups by several eria, and the pattern of a scanned letter is compared with ndard patterns. It was found that the letter could be identified hin a tolerable margin by associating it with the standard pattern h the smallest number of lack of coincidences with it.

Model for the Transmission of Speech by Recognition (I; Novem-11, 1960, and A; December 8, 1960)—G. Suzuki and K. Nagata.

A simple preliminary model of an efficient speech-transmission tem using recognition is presented. This model is limited to the nsmission of the "phonetic quality" of speech, *i.e.*, the information necessary to identify a speech sound as a linguistic code, and not the "vocal quality," which provides information regarding emotion and personality. The model can recognize vowels in C-V-type syllables, code the decision into teletype signals for transmission, and reproduce the vowels at the receiving end by a speech synthesizer. The recognition scheme is simply based on a frequency analysis of the input speech wave; the envelope of the input speech provides supplemental information on timing. A series of recognition tests reveals that this simple model can recognize vowels in C-V-type syllables with an accuracy of 100–80 per cent for a single male speaker, and has an average score of about 70–50 per cent for a group of 5–9 male speakers.

On the Number of Types of Self-Dual Logical Functions (I; December 16, 1960)—I. Toda.

Formulas for the number of self-dual logical functions of n variables and for the number of their symmetry types are derived with the aid of a modified Slepian method. The numbers are tabulated for the cases of six or less variables.

The Optimal Filter in the Phase-Locked FM Demodulator (I; January 17, 1961)—T. Tsumura and S. Kobayashi.

The optimal filter for a phase-locked FM demodulator is determined using Wiener's least-mean-square method and the following design criteria: 1) the noise of the signal due to noise interference should be minimized, and 2) the transient error between the output and the desired operation on the input, for a specific input, should be maintained at a specified level.

The following papers appear in the "Transactions of the Second Prague Conference on Information Theory, Statistical Decision Functions, and Random Processes (June 1–6, 1959)." They were published by the Publishing House of the Czechoslovak Academy of Sciences, Prague, Czechoslovakia, 1960. The affiliations of the authors are given below, as are some abstracts.

eakly Markov Queues—V. E. Beneš (in English). (Bell Telephone bs.)

A stochastic process $W(t)$ describes the length of time a customer uld have to wait in a queue if he arrived at time t. There is one ver, service is in order of arrival, and there are no defections. me mild assumptions of stationarity are formulated, and some neralized irrelevance or "weak Markov" conditions are described; s hoped that these will be helpful in problems other than queueing. e usefulness of the new approach is illustrated by showing that yields close general analogs of results previously known only for special case of Poisson arrivals and independent service times.

Random Solutions of Integral Equations in Banach Spaces—T. Bharucha-Reid (in English). (Univ. of Oregon and Math. t. of the Polish Acad. Sci., Wroclaw, Poland.)

Our purpose in this paper is to study some problems in what ght be called the theory of random operator equations. We begin s with the study of random operator equations in Banach spaces, h particular interest in the existence and measurability of the dom resolvent operator associated with a random operator. We n present a general discussion of the stochastic boundary value blem. After a brief discussion of Orlicz spaces and their properties, l generalized random variables with values in an Orlicz space, consider random Fredholm integral equations of the second d in Orlicz spaces. The existence of a random solution of the edholm integral equation is established. Finally, we discuss ults obtained for other Banach spaces and mention several er integral equations that are being studied within the framework probabilistic functional analysis.

ite-State Channels—L. Breiman (in English). (Univ. of Cali-nia, Los Angeles.)

Finite-state channels form an elegant and simple generalization zero-memory channels. The fundamental theorem of information theory (Shannon's theorem) has previously been proven for finite-state channels under the restriction that the channel be indecomposable. This condition, however, is quite restrictive and difficult to verify. In this paper we first redefine the channel capacity so that Shannon's theorem holds for the general finite-state channel. We then proceed to the question of when channels may be considered as being decomposable into subchannels, and give a simple answer. Finally, we derive a number of inequalities to facilitate the actual computation of channel capacity.

A Relative Limit Theorem for Parabolic Functions—J. L. Doob (in English). (Univ. Of Illinois.)

Convergence of Compact Measures on Metric Spaces—M. Driml (in English). (Czechoslovak Acad. Sci., Prague.)

On Experience Theory Problems—M. Driml and O. Hanš (in English). (Czechoslovak Acad. Sci., Prague.)

The present paper aims at a formulation of the typical experience theory problem, which is a natural consequence of the detailed study of common features occurring in many special cases. Although we do not propose a clear-cut definition of experience theory, we emphasize that its object is to improve gradually our decision procedure by utilizing past experience obtained from results of experimentation and observation. Four theorems are stated in which a special case dealing with continuous time is solved. Three of them assume delay in construction of the decision process, and the fourth works without any delay.

Continuous Stochastic Approximations—M. Driml and O. Hanš (in English). (Czechoslovak Acad. Sci., Prague.)

Roughly speaking, this paper deals with a continuous stochastic approximation method or, a little more precisely, with the continuous and probabilistic analog of the classical fixed-point theorem for

separable Banach spaces. The most important feature of the original Robbins-Munro stochastic approximation process can be expressed as follows: at each time instant a single experiment is performed, the level of which has been determined previously on the basis of prior outcomes only. Considering this feature, we aim at defining such a procedure which approximates continuously the fixed point of the expected value of a stationary ergodic stochastic process with values in a separable Banach space, utilizing at each instant only past history with a constant positive delay for the choice of level. The result, together with its discrete analog, is given.

Conditional Expectations for Generalized Random Variables—M. Driml and O. Hanš (in English). (Czechoslovak Acad. Sci., Prague.)

Stochastic Approximations for Continuous Random Processes—M. Driml and J. Nedoma (in English). (Czechoslovak Acad. Sci., Prague.)

The theory of stochastic approximations was founded by H. Robbins and S. Munro. The main problem considered by this theory is to find some characteristic point (namely, the zero point or point of minimum value) of the regression function of a one-parameter system of random variables. The regression function is assumed to be unknown and the characteristic point is determined by sequential approximation in such a way that random samples are taken from populations with distribution functions, the parameter of which is based on results of foregoing samples. The approximations of the characteristic point are obtained step by step.

The question of the extension of the method of stochastic approximations to the continuous cases arises. There are different possibilities of how to define continuous stochastic approximations. However, the existence of analog computers is a reason for seeking a method which enables the use of these computers; such a method is discussed in this paper.

Problems of Statistics Related to Markov Processes—R. M. Fortet (in French). (Univ. of Paris.)

The following general problem is considered in the special case of a Markov process. Being given a random function $X(t)$, defined on an interval of time $(0, t)$, the observation which would obtain the maximum possible information would be the continuous observation of $X(t)$ on $(0, t)$; but very often it would be easier to proceed with a periodic discrete observation on $X(t)$, of period T, at the instants $0, T, 2T, \cdots, kT, \cdots, (n-1)T$ (with $nT = t$); it is then interesting to evaluate the loss of information which is entailed in such a periodic discrete observation with respect to a continuous observation.

On a Problem in the Theory of Queueing—B. V. Gnedenko (in Russian). (Ukrainian Acad. Sci., Kiev.)

Up to the present time, as far as the author knows, the possibility of the dropping out of the operating state of the serving equipments has not been considered in queueing theory. In the present paper, we consider one of the basic problems in queueing theory with regard for this possibility. Consideration is limited to the case where a demand, finding all equipments taken or in nonworking condition, quickly disappears. Two cases are investigated: 1) if an equipment fails during a time of service, the demand being serviced disappears even under the condition that there are other free equipments, 2) if an equipment fails during a time of service, but there is a free equipment, the demand from the equipment which failed is transferred to a free equipment and service continues. The probability that k equipments are serving demands at instant t is calculated in these two cases, and the limit of this probability as $t \to \infty$ is evaluated.

On a Simple Linear Model in Gaussian Processes—J. Hájek (in English). (Czechoslovak Acad. Sci., Prague.)

This paper contains 1) proof of the existence of a random variable which is a sufficient statistic for the linear model, 2) a criterion for the regular case, and 3) a method of finding in the regular case the sufficient statistic mentioned in 1). These results are applied to processes with independent increments, to Markov processes and to stationary processes.

An Elementary Convergence Theorem—O. Hanš (in English). (Czechoslovak Acad. Sci., Prague.)

Random Fixed Point Approximation by Differentiable Trajectories—O. Hanš and A. Špaček (in English). (Czechoslovak Acad. Sci., Prague.)

The Entropy of the Swedish Language—H. Hansson (in English). (Tel. AB. L. M. Ericsson, Stockholm.)

Two methods suggested by C. Shannon have been used to determine the entropy of the Swedish language. The results are in rather good accordance with those obtained by others for English and German.

An Electronic Generator of Random Sequences—J. Havel (in English). (Czechoslovak Acad. Sci., Prague.)

In this paper an electronic generator of random sequences is described. First, the basic principle of the source of the random process and its transformation into a binary sequence of pulses with probability $\frac{1}{2} - \frac{1}{2}$ is explained. Then the generator itself is described and block diagrams are given. Also presented is a description of a unit for converting the waveform into a continuous stationary Gaussian process.

On the Capacity of Periodic and Almost-Periodic Channels—K. Jacobs (in German). (Univ. of Göttingen.)

This paper concerns itself with the so-called coding theorem. The Khinchin coding theorem for stationary channels is explained, and the Khinchin proof analyzed. It is then shown that for periodic and almost-periodic channels a corresponding coding theorem is obtained.

Explicit Formulas for the Extrapolation, Filtering and Computation of Information Content in the Theory of Gaussian Stochastic Processes—A. M. Yaglom (in Russian). (Acad. Sci. USSR, Moscow.)

A survey is given of some recent investigations related to two fields of the theory of probability—the theory of extrapolation and filtering, and the theory of information. Such a union of two apparently diverse fields is shown to have a definite basis, rather than being merely an artifice.

Some Properties of Markov Chains Added Modulo k—Z. Koutský (in German). (Czechoslovak Acad. Sci., Prague.)

Let a random variable be defined as the sum modulo k of the first n elements in a Markov chain. The properties of this random variable are studied; in particular, necessary and sufficient conditions are given, that as $n \to \infty$ all k values that this random variable may assume now become equally probable.

Necessary Convergence Conditions for Martingales and Related Processes—K Krickeberg (in German). (Univ. of Heidelberg.)

On a Characterization of the Wiener Process—R. G. Laha and E. Lukacs (in English). (Catholic Univ. of America, Washington, D. C.)

The following theorem is proved. Let $X(t)$ be a stochastic process defined in a finite closed interval $[A, B]$, and let the process be homogeneous with independent increments, and of second order with mean value function and covariance function of bounded variation in $[A, B]$. Let $a(t)$ and $b(t)$ be two continuous functions defined in $[A, B]$ such that $a(t)b(t) \neq 0$ for all $t \in [A_1, B_1]$ where $A \leq A_1 < B_1 \leq B$, and suppose $a(t)$ is not proportional to $b(t)$. Further, let $Y = \int_A^B a(t)\, dX(t)$ and $Z = \int_A^B b(t)\, dX(t)$ be two stochastic integrals, defined as limits in the mean. Then process $X(t)$ is a Wiener process if, and only if, 1) Y has linear regression on Z, and 2) the conditional variance of Y, given Z, does not depend on Z.

ı Some Connections of the Information Quantities of C. Shannon d R. Fisher with the Theory of Summation of Random Vectors—ı. V. Linnik (in Russian). (Univ. of Leningrad.)

In the present work, some connections are established between e two concepts of quantity of information of Shannon and Fisher. ith their help, a purely information-theoretic proof is successlly constructed of the central limit theorem for random vectors der the Lindeberg condition.

ıe Limit Properties of the Probability Distributions of Bounded arkov Processes—P. Mandl (in French). (Czechoslovak Acad. i., Prague.)

The present work contains the results of a study of the approach the stationary state of the homogeneous Markov processes scribing diffusion bounded by one or two barriers. Different types barriers are considered. The particle which arrives at the barrier ay be either absorbed or reflected or there may be an elastic rrier. Also presented is a theorem concerning diffusion without undaries.

n Generalized Stochastic Processes—G. Marinescu (in French). omanian Acad. Sci., Bucharest.)

n Measure Theory in Product Spaces—K. Matthes (in German). Iumboldt Univ., Berlin.)

n Nonergodic Channels—J. Nedoma (in English). (Czechoslovak cad. Sci., Prague.)

Several different definitions of the capacity of a channel have peared. It can be defined as the upper bound of the number of urces which are transmissable through the channel with arbitrarily nall probability of error; this is the e capacity of the channel. ıe capacity may also be defined as the upper bound of information tes which are obtained for all sources on the input space of the annel; this is the R capacity. Shannon's theorem may be proved r both these capacities, but the class of channels for which it lds is more restricted for the latter definition. On the other nd, the first part of Shannon's theorem can be proved for all godic sources with entropy rate less than the upper bound of formation rates obtained for those sources on the input of the annel which give an ergodic source-channel probability; this pper bound is called the ER capacity.

The question of the relationship of these three capacities arises. om the definition of ER capacity and R capacity, it follows ımediately that these are equal for all channels which are ergodic the sense that for all ergodic sources the source-channel probility is ergodic. Also, for a wide class of channels the ER capacity less than or equal to the e capacity, which is less than or equal the R capacity. Consequently, for ergodic channels all three pacities are equal.

The aim of this paper is to analyze the validity of such relations r nonergodic channels.

symptotically Stationary Gaussian Random Processes Produced Filtering of a Periodic Sequence of Pulses—C. Pantelopoulos ı French). (Czechoslovak Acad. Sci., Prague.)

Let $\{R(t, \tau)\}$ be a class of impulse responses of linear filters, here τ is a time constant. Conditions are given under which the ıss of output processes of these filters in response to a periodid quence of random, equiprobable, ± 1 rectangular pulses, converges a stationary Gaussian process as $\tau \to \infty$.

n Information Theory and Discernability in Statistical Decision oblems—A. Perez (in French). (Czechoslovak Acad. Sci., Prague.)

This work is concerned with the general problem of the transissability of an information source through a communication annel in the case of abstract alphabets, the time parameter being ther discrete or continuous. The probabilistic concept of "discernbility," which serves as a starting point for the concept of "transissability," is here enriched by its fusion with the idea of generalized k from the theory of statistical decision functions. After a brief scussion of the classical model of statistical decision, the concept discernability in decision problems is considered, both when there is and when there is not coding. Then the concept of transmissability is introduced, which is followed by consideration of certain theorems in information theory from the viewpoint of transmissability.

Experience and the Information Drawn from it with the Aid of the Limit Laws of Probability Theory—A. Perez (in French). (Czechoslovak Acad. Sci., Prague.)

On the Spreading Process—A. Prékopa (in English). (Hungarian Acad. Sci., Budapest.)

Suppose that in an abstract set a random point distribution of the Poisson type is given, and suppose that each random point generates a further random point distribution in the same space. Such a process is realized by the propagation of plants on the plane when the wind carries away the seeds. We thus have a time process of random point distribution. Such a process is called a spreading process, and is studied here.

An Effective Method of Finding Bayes' Solution—V. S. Pugachev (in Russian). (Acad. Sci. USSR, Moscow.)

A general method of finding Bayes' solution is given for an arbitrary loss function in the case when the random function being observed and estimated depends on a finite-dimensional random vector U, and the conditional distribution of the observed random function for any fixed value of the vector U is normal. This method gives the possibility under highly general conditions of finding optimal systems intended for the detection and reproduction of signals in the presence of additive noise. In particular cases, the method set forth yields earlier known methods of the determination of optimal systems for the case of additive normal noise.

On the Existence of Entropy—C. Rajski (in English). (Polish Acad. Sci., Warsaw.)

Let ξ denote a random variable defined in R_1, $f(x)$ its probability density function and $H(\xi)$ its entropy. The following theorem is proved: The entropy exists provided 1) the pdf exists and is monotonic except for a finite interval, and 2) there exists such a positive number ϵ that the integral $\int_{-\infty}^{\infty} | x |^{\epsilon} f(x)\, dx$ converges.

The Pseudometric Space of Discrete Random Variables Defined Over a Group—C. Rajski (in English). (Polish Acad. Sci., Warsaw.)

A proof is given that any set of all discrete random variables having a common sample space G is a pseudometric space with distance $d(\xi, \eta) = H(\xi - \eta)$ provided G is a group. A pseudometric space differs from a metric space only in the respect that $d = 0$ does not imply $\xi = \eta$.

Dimension, Entropy and Information—A. Rényi (in English). (Hungarian Acad. Sci., Budapest.)

On Optimal Multistage Tests—H. Richter (in German). (Univ. of Munich.)

Normalized ϵ-Entropy of Sets and the Theory of Transmission of Information—M. Rosenblatt-Roth (in Russian). (Parkhon Univ., Bucharest.)

The author has previously studied nonstationary (or, as a special case, stationary) stochastic sources and channels with arbitrary sets of states, time being considered discrete. In the case where the sources and channel inputs possess discrete sets of states, but the output sets of states of the channels are arbitrary, the fundamental two theorems of Shannon were proved for regular sources and channels.

In this work the question is posed of the approximation of stochastic nonstationary (or stationary) sources possessing continuous sets of states with stochastic sources possessing discrete sets of states, and also the question of the approximation of nonstationary (or stationary) stochastic channels possessing continuous sets of states at the input by stochastic channels possessing discrete sets of states at the input. The fundamental theorems of Shannon are considered in these conditions.

Relationships Between Information Theory and Decision-Function Theory—J. Seidler (in English). (Polish Acad. Sci., Gdansk, Poland.)

The relations between information theory and decision-function theory are considered. A decision is called a type-1 decision if it is an estimator and a type-2 decision if it is an estimating subset. The concept of a first-stage transformation of the received signals before making a decision is introduced. Formulas of the Rao-Cramer type for decisions of the first and second type corresponding to the Bayes method are derived. General conclusions concerning applications of entropy and amount of information if coding is not considered are given. Finally, a new uniqueness theorem for the entropy functional is proved.

Some Functionals in Processes—V. Statuliavichus (in Russian). (Lithuanian Acad. Sci., Vilnius.)

The applicability of the theorem of large deviations to nonhomogeneous Markov chains with a finite number of possible states is studied, as are sequences of chains, the nth chain being one with n instants of time.

Filters and Predictors which Adapt Their Values to the Unknown Parameters of the Input Process—O. Šefl (in English). (Czechoslovak Acad. Sci., Prague.)

This note is connected with the idea of a self-optimizing predictor which was described by L. Prouza in 1956. Prouza described a discontinuous predictor which adapts its parameters according to the measured coefficients of the input process. The main aim of this paper is to develop this idea for a continuous predictor which continuously adapts its characteristic to the correlation function of the input process.

Statistical Estimation of Provability in Boolean Logic—A. Špaček (in English). (Czechoslovak Acad. Sci., Prague.)

Random Metric Spaces—A. Špaček (in English). (Czechcslovak Acad. Sci., Prague.)

An abstract set together with a distance function defined for all pairs of its elements is said to be a rigid metric space. In view of various applications (in particular, in the fields of information theory and statistical decision processes, to the probabilistic concept of discernability of A. Perez), it is reasonable to replace the rigid metric by a random metric in order to obtain a random metric space. The properties of such spaces are considered here.

Random Mikusinski Operators—M. Ullrich (in English). (Czechoslovak Acad. Sci., Prague.)

After a discussion of the generalization of ordinary Mikusinski operators to the multidimensional case, a definition of a random Mikusinski operator is given, and its relation to ordinary random variables and stochastic processes shown. Then some new definitions, for random operators, of notions similar to those used in the theory of stochastic processes are given, and some fundamental properties of random Mikusinski operators are proved. Finally, the notion of a random operator function is introduced, and this is used for the solution of random partial differential equations.

A Representation Theorem for Random Schwartz Operators—M. Ullrich (in English). (Czechoslovak Acad. Sci., Prague.)

A Contribution to the Theory of Generalized Stationary Random Fields—K. Urbanik (in English). (Polish Acad. Sci., Wroclaw, Poland.)

Communication Channels with Finite Past History—K. Winkelbauer (in English). (Czechoslovak Acad. Sci., Prague.)

In this paper, basic theorems which are valid for channels with finite past history are given, namely a theorem of the type of the Feinstein-Khinchin fundamental lemma on discernability (the Coding theorem) and its converse, the direct and converse parts of a theorem of the Shannon type for transmission with arbitrarily small probabilities of error, theorems of the Shannon type for equivocation, the direct and converse parts of a theorem for transmission with arbitrarily small average frequencies of errors, theorems on transmission with arbitrarily small risks with respect to general weight functions, a theorem on transmission in the case of equality between entropy and capacity, and consequences of the latter theorems for the special case of channels with finite memory and, more generally, for the case of indecomposable channels.

Fundamental Equations of the Theory of Pursuit—A. Zieba (in English). (Polish Acad. Sci., Wroclaw, Poland.)

The minimax solution of the general problem of pursuit in the plane is presented for the case of one pursuer and one escaper. The result may be generalized for more than one pursuer and/or escaper as well as for pursuit in n-dimensional space.

On Certain Infinitesimal Properties of Random Functions—F. Zítek (in French). (Czechoslovak Acad. Sci., Prague.)

In this paper, certain properties of random functions of interval, such as absolute continuity and differentiability, are studied, as well as their relationships with the theory of limit laws and the theory of stochastic differential equations.

3ook Reviews

Statistical Theory of Communication—Y. W. Lee
Iohn Wiley and Sons, Inc., New York, N. Y.; 1960.
)9 Pages. $16.75)

Dr. Lee's book is an excellent self-integrated book on ie first principles of statistical theory of communication, hich in the author's usage, means linear least-mean-quare filtering and related subjects. The level of material ; accurately stated in the preface to be that of first-year raduate students or advanced seniors. It is presently eing used at M. I. T. in a first-year graduate class, and hould be quite understandable and easy to use in various eminars.

Material such as is presented in this book is absolutely ssential to a clear understanding of closely refined com-unication and tracking systems operating near maximum fficiency; *i.e.*, operating near threshold. It is a particularly ood introduction to statistical communication in that ; will equip the reader with the ability to attack more ifficult material later on.

The book should be fairly easy for communication ngineers to read, as the material is motivated by Lee's iterest in the filtering of noise from communication ignals. His experimental results contained in the latter hapters are excellent pieces of motivation.

It is only fair to warn the prospective teacher or student hat even the elements of statistical communication theory re by no means simple to understand thoroughly. The tudent must be used to thinking in integral and dif-erential equations. The book averages four or five equa-ions per page and these equations are used as part of he normal progression of text material. In other words, skipping over the equations without understanding them would be like skipping every other paragraph in a novel.

A teacher will probably find it necessary to make up problems for Chapters 3–5, 10, 11, 18, and 19 in order to fix in the student's mind the exact relationships involved in probability theory. The material on calculus of variations definitely should be supplemented either by other course work or by special notes, inasmuch as certain communications problems are particularly well handled by the application of the calculus of variations to the Weiner optimization.

The book should also prove useful to engineers confronted with the measurement problem of statistical data. Chapters 11 and 12 of the book are particularly interesting in showing that the measurement of statistical data produces answers which are only probabilistic in nature. The book shows how to treat measurement problems with theory that can be mastered by most instrumentation engineers. The last two chapters (18 and 19) describe some work with orthogonal functions and their generation with simple linear networks. This approach is particularly interesting because it shows simple ideas which should have good application of the representation of signals through orthogonal functions. Chapter 13 on the transfer characteristic of linear systems has proven quite valuable in advanced development work in the communications field.

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Papers can be of two kinds, tutorial or research, and should be so indicated. The former must be well-written expositions summarizing the state of a field in which research is still in progress, or else unify results scattered in the literature. Research papers must be original contributions not published elsewhere. They must present new methods, concepts, or ideas, or extend old ones to new areas of applicability; or, they must present new data, findings or inventions, or solve new problems of more than casual interest. They will not be accepted if, in the view of the reviewers and editors, they constitute a straightforward and easy application of existing theory to a special case of limited interest. It is not necessary that the length of each research paper be great; on the contrary, the submission of short but formal research notes is to be encouraged.

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